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ON TRANSLATIVITY OF ABSOLUTE ROGOSINSKY-BERNSTEIN METHODS OF DIFFERENT ORDER*

В статье «О преобразовании абсолютных методов Рогозинского - Бернштейна различного порядка» рассмотрена проблема преобразования абсолютного метода суммируемости Рогозинского - Бернштейна $(B_{h,r})$ различных порядков h, r;

$h-r\in[1/2, 1]$. Установлено, что когда $1/2<h-r\leq 1$ и $h-r=1/2$, то $|B_{h,r}|$ может быть преобразовано.

A series $a_0+a_1+a_2+ \ldots$ of real or complex terms a_n with its partial sums s^n

method $(B_{h,r})$ of order h and r; $h-r\in[0, 1]$, if $t_n\to t$ as $n\to\infty$, where

$$(B_{h,r}) \qquad t_n=\sum_{k=0}^{n}\left[\cos\frac{(k+r)\pi}{2(n+h)}\right]a_k. \tag{1}$$

The series is evaluable to M by the Ceasa'ro mean of (C, 1), if $M_n\to M$ as $n\to\infty$, where

$$(C, 1) \quad M_n=\frac{1}{n+1}\sum_{k=0}^{n}S_k. \tag{2}$$

The series is evaluable to g by the method (N), if $g_n\to g$ as $n\to\infty$, where

$$(N) \quad g_n=\frac{s_0+s_2+\ldots+s_{n-1}}{n+1}+\frac{s_n}{2(n+1)}. \tag{3}$$

The series is absolutely summable $(B_{h,r})$ or summable $|B_{h,r}|$ if the sequence $\{t_n\}$ is of bounded variation; that is to say, if

$$\sum_{n=0}^{\infty}|t_n-t_{n-1}|=O(1); \quad t_{n-1}=0. \tag{4}$$

Similar definition for the series being absolutely summable (C, 1) or (N).

A sequence-to-sequence method (A) is called translative to the left, if the limitability of s_1, s_2 . s_n, implies the limitability of $0, s_1, \ldots, s_{n-1}$, to the same limit, it is translative to the right, if the converse holds. (A) is translative if and only if, (A) is translative to the left and right.

On translativity much work have been done already; see [2, 3, 6, 8, 12, 13]. The special case in which $r=0$, the method given in (1) reduces to a well-known method (B_h) which had been the subject of the papers published in [1, 9-11, 16], and on absolute method of (B_h) was the subject of the papers by Azmi Al-Madi; see [4-6].

* Текст статьи публикуется в авторской редакции.

1. Object of the paper

In [8; Theorem (4.1)], Al-Madi investigated the translativity of $|(B_h)|$; $1/2 \le h \le 1$. In this paper we will prove analogous result to Al-Madi [8] for $|B_{h,\,r}|$; $1/2 \le h - r \le 1$. These results will be more general than that done by Al-Madi [8].

2. Subsidiary results

This section is devoted to results that are necessary for our proposes:

Theorem 2.1 ([7, Theorem (3.2)]). *If* h-r>1/2, *then* $|B_{h,\,r}|$ *and* $|C,\,1|$ *are equivalent.*

Theorem 2.2 (Mears [14]). *The sequence-to-sequence transformation*

$$t_n = \sum_{k=0}^{\infty} A_{n,\,k}\, S_k \tag{5}$$

is absolutely regular if and only if

$$\sum_{k=0}^{\infty} A_{n,\,k} \text{ converges for every } n, \tag{6}$$

and

$$\sum_{n=0}^{\infty} \left| \sum_{u=k}^{\infty} A_{n,\,u} - \sum_{u=k}^{\infty} A_{n+1,\,u} \right| = O(1). \tag{7}$$

Theorem 2.3 (Parameswaran [15, Theorem (4)]). *If the sequence-to-sequence transformation given by* (5) *is regular and absolutely regular, and if*

$$\left|A_{n,\,n}\right| - \sum_{k=n+1}^{\infty} \left| \sum_{i=n}^{k} A_{k,\,i} - A_{n-1,\,i} \right| > \lambda > 0, \tag{8}$$

for all n; n=0, 1, 2,.... *then the transformation given in* (5) *is equivalent to absolutely ineffective.*

3. Main results

In this section we present and prove our main results:

Theorem 3.1 *If* $1/2 < h - r \le 1$, *then* $|B_{h,\,r}|$ *is* translative.

Theorem 3.2 *If* h-r=1/2, *then* $|B_{h,\,r}|$ *is translative.*

Proof of theorem 3.1. Let $\{M_n\}$, $\{\bar{M}_n\}$ be respectively the (C, 1) transforms of $\{S_n\}$, $\{S_{n-1}\}$. Using (2) to find $\bar{M}_{n+1}$ in terms of M_n. The result is

$$\bar{M}_{n+1} = \sum_{k=0} \alpha_{n,\,k} M_k, \tag{9}$$

where

$$\alpha_{n,\,n} = \frac{n+1}{n+2}, \tag{10}$$

$$\alpha_{n,\,k} = 0 \text{ otherwise.} \tag{11}$$

Using (10) and (11), one can easily checked that (6) - (8) are all satisfied, and theorems (2.2) and (2.3) imply that $|C,\,1|$ is translative, and theorem (2.1) implies result.

Proof of theorem 3.2. To obtain the result, it is enough to show that $|N|$ is translative and that when h-r=\l2, then $|B_{h,\,r}|$ and $|N|$ are equivalent.

Write (3) in the form

$$g_n = \sum_{k=0}^{\infty} \beta_{n,\,k}\, S_k, \tag{12}$$

where

$$\beta_{n,\,n} = \frac{1}{2(n+1)}, \tag{13}$$

$$\beta_{n,\,k} = \frac{1}{n+1}; \quad 0 \le k \le n-1 \tag{14}$$

and

$$\beta_{n,\,k} = 0 \text{ otherwise.} \tag{15}$$

One can easily checked (6) and (7) are satisfied and theorem (2.2) implies that (N) is absolutely regular. Next, assume that $\{g_n\}$ and $\{\bar{g}_n\}$ be respectively the (N) transforms of $\{S_n\}$ and $\{S_{n-1}\}$. Using (3), we have

$$\bar{g}_{n+1} = \sum_{k=0}^{\infty} b_{n,k}\, g_k, \tag{16}$$

where

$$b_{n,n} = \frac{n+1}{n+2}, \tag{17}$$

$$b_{n,k}=0;\ k \neq n, \tag{18}$$

condition (8) follows at once from (17) and (18), and theorem (2.3) implies that $|N|$ is translative.

Finally, to show that when $h-r=1/2$, $|B_{h,r}|$ and $|N|$ are equivalent, we write

$$a_k = S_k - S_{k-1} = (k+1)M_k - 2kM_{k-1} + (k-1)M_{k-2}, \tag{19}$$

where

$$S_j = M_j = 0;\quad j = -1, -2, \ldots, \tag{20}$$

$$g_n = \frac{1}{2(n+1)}\left[nM_{n-1} + (n+1)M_n\right], \tag{21}$$

and

$$\frac{\pi}{4(n+h)} = \theta_n, \tag{22}$$

to put (1) in the form

$$t_n = \sum_{k=0}^{n} F_{n,k}\, g_k, \tag{23}$$

$$F_{n,n} = 2(n+1)\sin\theta_n, \tag{24}$$

where

$$F_{n,n} = -4(k+1)\sec\theta_n \sin^2\theta_n \cos(2k+2r+1)\theta_n;\quad 0 \le k \le n-1, \tag{25}$$

$$F_{n,k} = 0 \quad \text{otherwise.} \tag{26}$$

The result would follow from theorem (2.2), (26) and theorem (2.3) if we show that the conditions (6) - (8) are all satisfied.

Put F

$$\begin{aligned} F_{n,k} &= -2\cos 2r\theta_n \sec\theta_n \sin^2\theta_n \left\{\frac{d}{d\theta_n}\sin(2k+1)\theta_n + \cos(2k+1)\theta_n\right\} + \\ &+ 2\sin 2r\theta_n \sec\theta_n \sin^2\theta_n \left[-\frac{d}{d\theta_n}\cos(2k+1)\theta_n + \sin(2k+1)\theta_n\right], \end{aligned} \tag{27}$$

and using the facts that

$$\sum_{k=0}^{n-1}\sin(2k+1)\theta_n = \frac{\sin^2 n\theta_n}{\sin\theta_n} \tag{28}$$

$$\sum_{k=0}^{n-1}\cos(2k+1)\theta_n = \frac{\sin 2n\theta_n}{2\sin\theta_n} = \tag{29}$$

and

$$= \frac{\cos\theta_n}{2\sin\theta_n}, \tag{30}$$

it follows from (24) - (30) that

$$\sum_{k=0}^{n} F_{n,k} = 2(n+1)\sin\theta_n - 2\sec\theta_n \sin^2\theta_n \left\{\cos 2r\theta_n \left[\frac{d}{d\theta_n}\frac{\sin^2\theta_n}{\sin\theta_n} + \frac{\cos\theta_n}{2\sin\theta_n}\right] + \right. \tag{31}$$

$$\left. + \sin 2r\,\theta_n \left[\frac{d}{d\theta_n}\frac{\cos\theta_n}{2\sin\theta_n} + \frac{\sin^2 n\theta_n}{\sin\theta_n}\right]\right\},$$

$$\sum_{k=0}^{n} F_{n,k} \to 1 \text{ as } n\to\infty \qquad (32)^*$$

and condition (6) is satisfied.

As for (7) use (24) - (26), we see that the left hand side of (7) is equal to

$$\gamma_k + \sum_{n=k+1}^{\infty} \left|D_{n,k} - D_{n+1,k}\right|, \tag{33}$$

where γ_k is bounded for all k, and

$$D_{n,k} = \sum_{u=k}^{n} F_{n,u} = \sum_{u=0}^{n} F_{n,u} - \sum_{u=0}^{k-1} F_{n,u} = \tag{34}$$

$$= 1 + (2k+1)\sec\theta_n \cos 2kr\,\theta_n \sin\theta_n \sin 2k\,\theta_n - 2\cos 2kr\,\theta_n \sin^2 k\,\theta_n = \tag{35}$$

$$= f_k(\theta_n), \text{ say.} \tag{36}$$

This shows that $D_{n,k}$ is of bounded variation in k, uniformly in $n \geq k+1$, and thus (7) is satisfied.

Finally, using (24) and (36), we see that the left hand side of (8) is equal to

$$2(n+1)\sin\theta_n - \sum_{k=n+1}^{\infty} \left|f_n(\theta_k) - f_n(\theta_{k-1})\right|. \tag{37}$$

Differentiate, and use the mean value theorem, we see that $f_n(\theta_k)$ is a decreasing function when κ increases for all $k \geq n+1$. Therefore (37) reduces to

$$2(n+1)\sin\theta_n + f_n(\theta_j) f_n(\theta_n); \quad j > n^2, \tag{38}$$

which by (35) reduces to

$$2(n+1)\sin\theta_n + (2n+1)\sec\theta_j \cos 2nr\theta_j \sin\theta_j \sin 2n\theta_j -$$

$$-2\cos 2nr\theta_j \sin^2 n\theta_j - (2n+1)\sec\theta_n \cos 2nr\theta_n \sin\theta_n \sin 2n\theta_n + \tag{39}$$

$$+2\cos 2nr\theta_j \sin^2 n\theta_n; \quad j > n^2,$$

$$\to \frac{\pi}{2} + \left(1 - \frac{\pi}{2}\right)\cos\frac{\pi r}{2} \geq \beta > 0, \tag{40}$$

for every value of r, and (8) is satisfied. This completes the proof.

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* We observe that if we write (23) in the form $t_n = \sum_{k=0}^{n} F_{n,k}(g_n - 1) + \sum_{k=0}^{n} F_{n,k}$ then in the special case in which $S_k=1$ all $k \geq 0$ implies $g \to 1$ and $t_n \to 1$ as $n\to\infty$ and (32) implies that $\sum_{k=0}^{n} F_{n,k} \to 1$ as $n\to\infty$.

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$$Pf(x,t) = \int_X p(x,y,t) f(y)\, d\mu(y)$$

$$\Gamma_\varepsilon(x_0) = \{(x,t) \in X \times (0,1] : d(x,x_0) < \varepsilon(t)\}. \qquad (1)$$

$\varepsilon(t)=t$

$\mathbb{R}^{n+1}_+$

$\mathbb{R}^n$

$\lim\limits_{t\to+0} t/\varepsilon(t)$

μ

μ

$\mu(B(x,t)) \approx t^\gamma (\gamma>0)$,

$d: X\times X \to [0,\infty)$

$$d(x,y)=0 \Longleftrightarrow x=y,\ d(x,y)=d(y,x),$$