# ASYMPTOTIC PROPERTIES OF BINARY SEQUENCES OBTAINED BY THE NEUMANN TRANSFORM

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**Abstract**

In this paper we investigate the properties of binary sequences consisting of non-identically distributed dependent elements obtained by the Neumann transform. Some results for the asymptotic behaviour of joint distribution of such sequences are obtained.

## 1 Introduction

In the beginning of 1950s John von Neumann proposed a simple method to transform the sequence of independent identically distributed binary random variables into the sequence of independent random binary variables taking values 1 and 0 with probabilities $\frac{1}{2}$. This method was used to improve the quality of physical random number generators.

Let $\xi_1, \xi_2, \dots$ be a sequence of binary random variables and

$$\tau_1 = \min\{k \geq 1 : \xi_{2k-1} \neq \xi_{2k}\},\ \tau_{n+1} = \min\{k > \tau_n : \xi_{2k-1} \neq \xi_{2k}\}, n = 1, 2 \dots \quad (1)$$

The Neumann transform $\{\eta_t\}_{t=1}^{\infty}$ of the binary sequence $\{\xi_t\}_{t=1}^{\infty}$ is defined by the following rule:

$$\eta_t = \xi_{2\tau_t - 1}, \quad t = 1, 2 \dots \quad (2)$$

In what follows we suppose that $\xi_1, \xi_2, \dots$ are independent and $\mathbf{P}\{\xi_i = 1\} = p > 0$, $\mathbf{P}\{\xi_i = 0\} = q > 0$, $p + q = 1$, then

$$\mathbf{P}\{\xi_{2k-1} = 0 \,|\, \xi_{2k-1} \neq \xi_{2k}\} = \mathbf{P}\{\xi_{2k-1} = 1 \,|\, \xi_{2k-1} \neq \xi_{2k}\} = \frac{1}{2},$$

and therefore $\eta_1, \eta_2, \dots$ are independent and $\mathbf{P}\{\eta_t = 0\} = \mathbf{P}\{\eta_t = 1\} = \frac{1}{2}$.
Let us consider the Neumann transform $\{\eta'_t\}_{t=1}^{\infty}$ of the shifted binary sequence $\{\xi_t\}_{t=2}^{\infty}$ defined as follows:

$$\eta'_t = \xi_{2\tau'_t}, \quad t = 1, 2 \dots, \text{ where} \quad (3)$$

$$\tau'_1 = \min\{k \geq 1 : \xi_{2k} \neq \xi_{2k+1}\},\ \tau'_{n+1} = \min\{k > \tau'_n : \xi_{2k} \neq \xi_{2k+1}\}, n = 1, 2 \dots \quad (4)$$

It is clear that the elements of the sequences $\{\eta_t\}_{t=1}^{\infty}$ and $\{\eta'_t\}_{t=1}^{\infty}$ are dependent and joint distribution of the elements of the sequences is not trivial. In this paper we investigate the asymptotic behaviour of distributions of a pair $(\eta_t, \eta'_t)$ and of vectors $(\eta_{t+1}, \dots, \eta_{t+l}; \eta'_{r+1}, \dots, \eta'_{r+s})$, as $t, r \to \infty$, where $l, s$ are any finite numbers.

## 2 Results

Let us consider the first elements of the sequences $\{\eta_t\}_{t=1}^{\infty}$ and $\{\eta'_t\}_{t=1}^{\infty}$.

**Lemma 1.** *If* $\mathbf{P}_{\alpha}^{\beta} = \mathbf{P}\{\eta_1 = \alpha, \eta'_1 = \beta\}$, $\alpha, \beta \in \{0,1\}$, *then* $\mathbf{P}_0^0 = \mathbf{P}_1^1 = pq/2$ *and* $\mathbf{P}_1^0 = \mathbf{P}_0^1 = \frac{q+p^2}{2}$.

The joint distribution of $(\eta_1, \eta'_1, \eta_2, \eta'_2)$ is more complicated. However the limit distribution of a pair $(\eta_t, \eta'_t)$ as $t \to \infty$ is very simple.

**Theorem 1.** *The elements* $\eta_t$ *and* $\eta'_t$ *of the sequences* $\{\eta_t\}_{t=1}^{\infty}$ *and* $\{\eta'_t\}_{t=1}^{\infty}$ *obtained by the Neumann transform (2) and (3) are asymptotically independent as* $t \to \infty$.

The proof of independency is based on the fact that the distribution of a pair $(\tau_t, \tau'_t)$ is asymtotically Gaussian. Then the relations

$$\left|\mathbf{P}\{(\eta_t, \eta'_t) = (\alpha, \alpha')\} - \frac{1}{4}\right| \leq \mathbf{P}\{|\tau_t - \tau'_t| < 2\} \to 0, \text{ when } t \to \infty, \text{ where } \alpha, \alpha' \in \{0,1\},$$

shows that the distribution of a pair $(\eta_t, \eta'_t)$ tends to the equiprobable one when $t \to \infty$.

The statement may be also applied to the sets of neighbouring elements of sequences $\{\eta_t\}_{t=1}^{\infty}$ and $\{\eta'_t\}_{t=1}^{\infty}$.

**Theorem 2.** *The elements* $(\eta_{t+1}, \ldots, \eta_{t+l}; \eta'_{r+1}, \ldots, \eta'_{r+s})$, $l, s \in \mathbb{N}$, *obtained from the sequences* $\{\eta_t\}_{t=1}^{\infty}$ *and* $\{\eta'_t\}_{t=1}^{\infty}$ *by Neumann transforms (2) and (3), are asymptotically independent at* $t, r \to \infty$.

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