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Analytical analysis of the "collapse-revival" effect in the Jaynes–Cummings model

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ABSTRACT

The evolution of the atomic state population in a two-level system coupled to a single-mode quantum field is calculated in the analytical form. Essential characteristics of the "collapse-revival" effect are expressed in terms of the physical parameters of the system by means of simple formulas in both the resonant and the non-resonant cases. The obtained results are of great importance for the qualitative analysis of the phenomenon.

1. Introduction

The Jaynes–Cummings model (JCM) [1] for the two-level atom strongly coupled to a one-mode quantum field in a cavity allows one to investigate a series of qualitatively new physical effects in the evolution of the atomic state population. In particular, the so-called "collapse-revival" effect (CRE) is of great interest as the pure quantum property of this system. It is important that CRE cannot be considered by means of the classical Bloch equations and was described theoretically for the first time on the basis of the strict calculation of the atomic density matrix by means of summation over the field states using the rotating wave approximation (RWA) (see, for example, [2]). The very first experimental observation of CRE was described in paper [3] and later this effect was also observed in other experiments (for example, [4]). At present it is actively discussed in quantum optics not only for atomic system applications [5] but also for the cases of low-dimension quantum nanostructures [6], the Landau–Zener process [7], and trapped atoms [8].

Generally, CRE means that the variation of the atomic population inversion $W(t)$ includes several qualitative properties: (i) the well-known fast oscillations (primary structure) that are defined by the Rabi period $T_R = 2\pi\Omega_R^{-1}$; (ii) the quasi-periodic oscillations with the period $T_C \gg T_R$, when the amplitudes of the fast oscillations turn to zero ("collapse") and then the system returns to the fast-oscillating regime again ("revival"); (iii) the amplitude of the Rabi oscillations decreases in every successive revival, and after some characteristic time $T_D \gg T_C$ the subsequent revivals overlap and $W(t)$ has no evident regular structure. However, it should also be mentioned that the fractional and super-revivals could appear on the long-range time scale [9].

As mentioned above, it is possible to describe these peculiarities of CRE numerically on the basis of JCM within RWA (see Fig. 1). However, it is difficult to extract the dependencies of the time scale characteristics of CRE on the physical parameters of the system (coupling constant, detuning, average field excitation) and to effectively control the evolution of the system using numerical analysis alone. The problem is complicated in addition beyond RWA when the energy spectrum of the system is changed drastically [10], and this case can be essential for the analytical analysis of CRE in the large field approximation [11]. Effective control of CRE is of great value when searching for the optimal parameters of quantum computers [12].

In many applications the average number of photons in a cavity is quite large. In this case the problem of the analytical description of CRE for the resonant condition was solved for the first time in [2] using the complex saddle point method. A similar calculation was fulfilled in [13] by means of the real saddle point approach together with the Poisson summation formula. In the present Letter, the analogous analytical analysis of CRE for such a system is ful-

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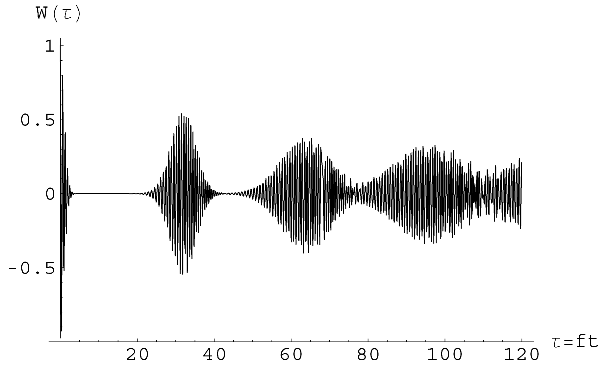

Fig. 1. Results of the numerical calculation of $W(t)$ for the resonant case with $|u|^2 = 25$.

filled using the complex saddle point when the subsequent revivals are conditioned by an appearance of the additional phases of the field amplitude because of the atom-field interaction during some time period. Simple analytical expressions are obtained that allow one to introduce various time scales and to evaluate all essential features of the effect for both the resonant and the non-resonant cases as well as to consider the field amplitude influence at CRE.

2. CRE for the resonant case

Let us use the well-known Hamiltonian of the JCM within the framework of RWA

$$\hat{H} = \frac{1}{2}E\hat{\sigma}_3 + \hat{a}^+\hat{a} + f(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^+). \tag{1}$$

Here the system of units with $\hbar = c = 1$ is used and the energy of a photon is chosen to be $\omega = 1$; E is the difference between the atomic energy levels, so that the resonant case corresponds to $E = 1$; f is the atom-field coupling constant; $\hat{a}^+$ and $\hat{a}$ are the photon creation and annihilation operators, respectively; and $\hat{\sigma}_3$ and $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_1 \pm \hat{\sigma}_2)$ are the Pauli matrices.

Within RWA the so-called "counter-rotating" terms $\hat{\sigma}_+\hat{a}^+$, $\hat{\sigma}_-\hat{a}$ of the full JCM Hamiltonian are omitted. The eigenfunctions and eigenvalues of the operator (1) are well known. This leads to the possibility of finding the time dependence of the population inversion in the form of the sum over the field quantum numbers [5]

$$W(t) = \rho_{\uparrow\uparrow}(t) - \rho_{\downarrow\downarrow}(t) = \sum_{n=0}^{\infty} e^{-|u|^2}\frac{|u|^{2n}}{n!}\cos\bigl[\Omega_R(n)t\bigr],$$
$$\Omega_R(n) = 2f\sqrt{n+1}. \tag{2}$$

It is assumed here that the quantum field is represented by the coherent state with amplitude u and the atom is in its upper state when the interaction is switched on ($t = 0$). The average number of photons and the dispersion of the initial state is defined by the well-known equations

$$\langle n\rangle = |u|^2, \qquad D_n = \langle n^2\rangle - \langle n\rangle^2 = |u|^2. \tag{3}$$

Fig. 1 shows the function $W(t)$ calculated numerically by evaluation of (2) for $|u|^2 = 25$. It demonstrates all the peculiarities of CRE mentioned above. The characteristic time scale used for the time axis is defined by the dimensionless parameter $\tau = ft$.

There is a method for the analytical estimation of the "collapse-revival" period T_C for $W(t)$ [5]. It is based on the expansion of the Rabi frequency near the value $\langle n\rangle$ with the accuracy of the first order in the parameter $\langle n\rangle^{-1} \ll 1$:

$$\Omega_R(n) \approx \Omega_R\bigl(\langle n\rangle\bigr)\left(1 + \frac{n - \langle n\rangle}{2(\langle n\rangle + 1)}\right). \tag{4}$$

As a result, the approximate analytical function $W_0(t)$ can be calculated, and it is the strict periodic one:

$$W_0(t + T_C) = W_0(t), \quad T_C \approx \frac{2\pi\sqrt{\langle n\rangle}}{f}. \tag{5}$$

Comparison of the functions $W(t)$ and $W_0(t)$ shows that the estimation (5) is quite good for T_C, but the real form of the revivals for $W(t)$ is essentially non-periodic.

Following paper [13], the approximate analytical calculation of (2) can be based on the replacement of summation by integration with the accuracy $\sim \langle n\rangle^{-1} \ll 1$. However, in the present Letter the obtained integral is estimated by the saddle point method in the complex plane [14] instead of by the Poisson summation formula used in [13]. Let us discuss this approach in detail.

With the considered accuracy, the value $n!$ can be replaced by the asymptotic Stirling formula and the function (2) is defined by the following integral ($n \to z$)

$$W(t) \approx W_A(t) = \frac{1}{\sqrt{2\pi}}\Re\left\{\int_0^\infty dz\, e^{\Phi(z)}\right\},$$
$$\Phi(z) = z\ln|u|^2 - |u|^2 + z - \left(z + \frac{1}{2}\right)\ln z + i\Omega_R(z)t,$$
$$\Omega_R(z) \approx 2f\sqrt{z}. \tag{6}$$

Analytical continuation of this integral to the complex plane

$$z = re^{i\varphi} \tag{7}$$

allows one to apply the saddle point method for its estimation [14]. The equation for the saddle point z_0 is given by

$$\Phi'(z_0) = -\ln\frac{z_0}{|u|^2} + i\frac{ft}{\sqrt{z_0}} = 0. \tag{8}$$

The function $\Phi(z)$ has a branch point that leads to the set of solutions for Eq. (8)

$$z_{0k} = r_k e^{i\varphi_k}, \quad \varphi_k = \varphi_0 + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

and the contribution of every saddle point should be taken into account in the estimation $W_A(t)$.

If one separates real and imaginary parts in Eq. (8), the following system of equations for the values r_k, φ_0 can be obtained:

$$\ln x_k = (-1)^k\frac{\tau}{|u|\sqrt{x_k}}\sin\frac{\varphi_0}{2}, \quad x_k \equiv \frac{r_k}{|u|^2}$$
$$\varphi_0 + 2\pi k = (-1)^k\frac{\tau}{|u|\sqrt{x_k}}\cos\frac{\varphi_0}{2}. \tag{10}$$

Eqs. (10) are solved in different ways for the cases $k = 0$ and $k \neq 0$. For the first stage of the evolution (damped Rabi oscillations) one can find the following estimations: ($k = 0$)

$$x_0 \approx 1 + \delta, \qquad \varphi_0 \ll 1,$$
$$\delta = \frac{1}{2}\frac{\tau^2}{|u|^2}, \qquad \varphi_0 \approx \sqrt{2\delta}. \tag{11}$$

The subsequent revivals are defined by the approximate solutions of Eqs. (10) with $k \neq 0$

$$x_k \approx 1 + 2\pi k\delta_k, \qquad \varphi_k \approx (-1)^k(2\pi k + 2\delta_k),$$
$$\delta_k = \frac{\tau/|u| - 2\pi k}{2 + \pi k\tau/|u|}, \quad k = 1, 2, \ldots. \tag{12}$$

The contribution of every solution to the integral (6) is defined by the usual saddle point evaluation

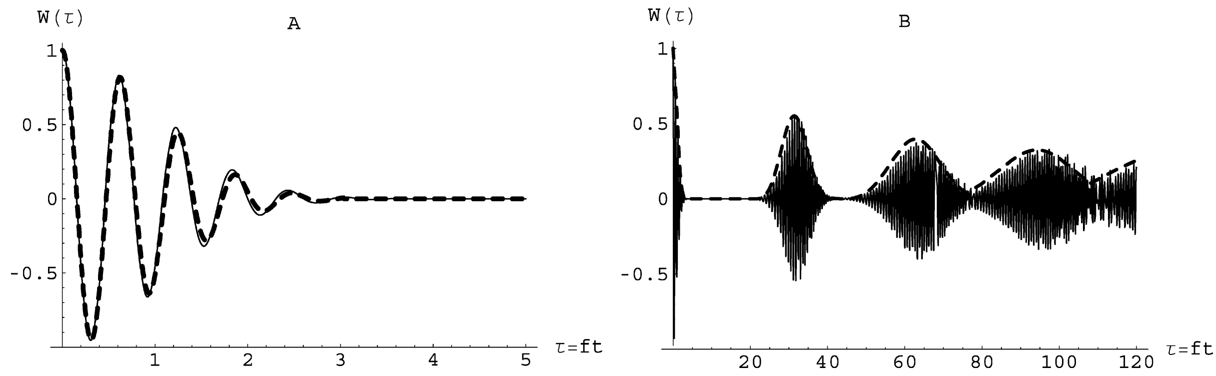

Fig. 2. Comparison of the results for $W(t)$ ($E = 1$, $|u|^2 = 25$) calculated numerically with Eq. (2) (solid line) and analytically with formula (13) (dashed line): (A) the high-frequency oscillations for small τ; (B) the analytical envelope for the wide time interval.

$$W_A^{(k)}(t) \approx M^{(k)}(\tau) R^{(k)}(\tau),$$
$$M^{(k)}(\tau) = \frac{1}{\sqrt{a}} e^{-|u|^2(1 - x_k\cos\varphi_k + x_k\cos\varphi_k \ln x_k - x_k\varphi_k \sin\varphi_k + 2\frac{\tau}{|u|}\sqrt{x_k}\sin\frac{\varphi_k}{2})},$$
$$R^{(k)}(\tau) = \cos\left[\frac{\pi - \alpha - \delta_k}{2} + |u|^2\left(x_k \sin\varphi_k - x_k\varphi_k \cos\varphi_k - x_k \sin\varphi_k \ln x_k + 2\frac{\tau}{|u|}\sqrt{x_k}\cos\frac{\varphi_k}{2}\right)\right], \tag{13}$$

where

$$a \equiv \left|-(1 - 2i\delta_k) - i\frac{\tau(-1)^k}{2|u|}\left((-1)^k - \pi k\delta_k - 3i\delta_k\right)\right|,$$
$$\alpha \equiv \arg\left(-(1 - 2i\delta_k) - i\frac{\tau(-1)^k}{2|u|}\left((-1)^k - \pi k\delta_k - 3i\delta_k\right)\right). \tag{14}$$

The high-frequency oscillations of the population inversion are described by the function $R^{(k)}(\tau)$, whereas its low-frequency part (envelope) is defined by the function $M^{(k)}(\tau)$. Analysis of Eq. (13) shows that for the arbitrary k the function $M^{(k)}(\tau)$ has a sharp maximum if the saddle point corresponds to the values $x_k \approx 1$; $\varphi_0 \approx 0$. Near this point the envelope function $M^{(k)}(\tau)$ can be approximated by the Gaussian form, which coincides with the result obtained in paper [13] with a slightly different definition of x_k. However, one should use the general formula (13) in order to describe the evolution between the revivals more precisely.

Thus the envelope is represented by the sum of the slightly overlapping peaks at the points $\tau_k = 2\pi k|u|$ so that the distance between them gives the period of the revivals:

$$T_C = 2\pi\frac{|u|}{f}. \tag{15}$$

The width of every subsequent revival increases as

$$\Delta_k t = \frac{1}{f}\sqrt{1 + \pi^2 k^2}, \tag{16}$$

and the fast oscillation amplitudes decrease $\sim M^{(k)}(\tau_k)$. The condition of the essential overlapping of the peaks defines the value T_D for CRE:

$$2\pi|u| = \sqrt{1 + \pi^2 k^2}, \quad T_D \approx \frac{4\pi|u|^2}{f}. \tag{17}$$

The analytical Eqs. (15)–(17) establish connections between all the essential characteristics of CRE and the physical parameters of the system. Fig. 2 shows that formula (13) describes both the high-frequency oscillations and the envelope of the population inversion with high accuracy.

It is important to note that if one considers the population inversion $W(T, |u|)$ as a function of the field amplitude $|u|$ with the fixed duration T of the atom-field interaction, the specific behaviour of the Rabi oscillations would appear as illustrated by Fig. 3. This should be taken into account when estimating the Rabi oscillations damping derived from the reservoir in an open system [15].

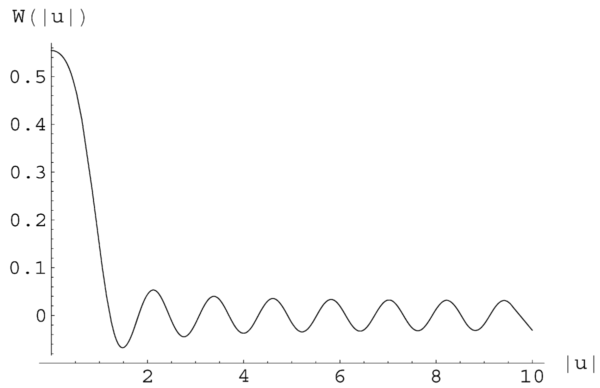

Fig. 3. Dependence of the population inversion on the field amplitude for $fT = 2.65$ and $E = 1$.

3. CRE for the non-resonant case

Let us now consider JCM with $E \neq 1$ (the value $(E - 1)$ defines the detuning) when the equation for the population inversion is given by:

$$W(t) = 1 - 2\sum_{n=0}^{\infty} e^{-|u|^2}\frac{|u|^{2n}}{n!}\frac{4f^2(n+1)}{\Omega_R^2(n)}\sin^2\left[\frac{1}{2}\Omega_R(n)t\right],$$
$$\Omega_R(n) = \sqrt{\Delta^2 + 4f^2(n+1)}, \quad \Delta = E - 1. \tag{18}$$

The general approach for the estimation of this sum is analogous to the one considered in the previous section. However it is important that the characteristic time scale $\tau_1 = t\sqrt{f^2 + (E-1)^2/|u|^2}$ changes for this case. Its dependence on the coupling constant turns out to be more complicated. Omitting all details of the calculations, let us show only the results of the analytical approximation for CRE.

$$x_k \approx 1 + 2\pi k\delta_k, \quad \varphi_k \approx 2\pi k + 2Q_k^2\delta_k,$$
$$\delta_k = \frac{2f^2 t/\Omega_R(|u|^2) - 2\pi k}{2Q_k^2 + ft\pi k/(Q_k^3|u|)}, \quad Q_k \equiv \frac{\Omega_R(|u|^2)}{2f|u|}, \quad k = 0, 1, 2, \ldots. \tag{19}$$

For the envelope, one can obtain the following formula:

$$M^{(k)} = 1 - \frac{1}{Q_k^2} + \frac{A}{\sqrt{B}} e^{-|u|^2(1 - x_k\cos\varphi_k(1 - \ln x_k) - x_k\varphi_k\sin\varphi_k - \Re\{it\frac{\Omega_R(z_{0k})}{|u|^2}\})},$$

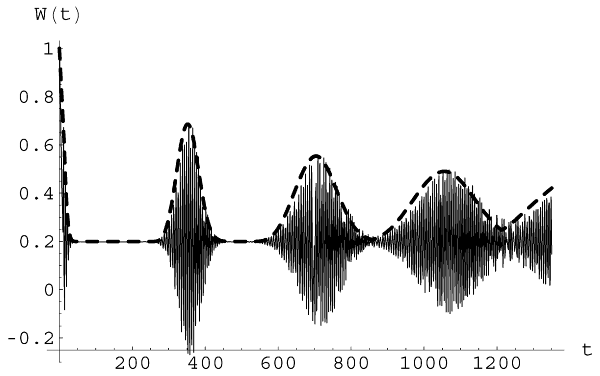

Fig. 4. Comparison of the results for $W(t)$ $(E = 1, 5;\ |u|^2 = 25;\ f = 0.1)$ obtained numerically by means of Eq. (18) (solid line) and the analytical envelope for the same function calculated with formula (19) (dashed line).

$$A \equiv \left|\frac{f^2|u|^2 x_k e^{i\varphi_k}}{\frac{\Delta^2}{4} + f^2|u|^2 x_k e^{i\varphi_k}}\right|,$$

$$B \equiv \left|1 + \frac{if^4 t}{2}\frac{|u|^2 x_k e^{i\varphi_k}}{(\frac{\Delta^2}{4} + f^2|u|^2 x_k e^{i\varphi_k})^{3/2}}\right|. \tag{20}$$

The following formulas for the period T_C and the widths of every subsequent revival can be derived

$$T_C = \frac{\pi\Omega_R}{f^2}, \qquad \Delta_k t = \frac{1}{f}\sqrt{1 + 16\frac{f^4|u|^4}{\Omega_R^4}\pi^2 k^2}. \tag{21}$$

The peak overlap condition leads to the estimation for T_D:

$$\frac{\pi\Omega_R}{f} = \sqrt{1 + 16\frac{f^4|u|^4}{\Omega_R^4}\pi^2 k^2}, \qquad T_D = \frac{\pi\Omega_R^4}{4f^5|u|^2}. \tag{22}$$

Fig. 4 shows that the analytical approximation for the non-resonant case also corresponds very well with the results calculated numerically.

In conclusion, it is interesting to note that the derived analytical approximation for CRE allows one to give a simple interpretation of the effect. The point is that the phase $\varphi(\tau)$ of the saddle point in formula (12) actually defines the adiabatic variation of the phase of the average field amplitude

$$\langle a\rangle \approx |u|e^{i\frac{\varphi(\tau)}{2}},$$

that appears due to the interaction of the quantum field with the two-level system. This interaction is most significant during the time intervals when the phase of the field is close to $2\pi k$ values. The latter results in the increase of the oscillations' amplitude for the population inversion and in the revival peaks as well.

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