ASYMPTOTIC SOLUTION OF THE PROBLEM OF OPTIMAL OBSERVATION OF A QUASILINEAR SYSTEM

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An alternate vertion asymptotic solution of the problem of optimal observation of the quasilinear dynamic systems is considered. The notation of s - optimal asymptotic solution is introduced and algorithm for the asymptotic solution of the problem of optimal observation is proposed.

KEY WORDS Quasilinear dynamic system, asymptotic solution, observation of initial state, linear semi-infinite programming.

1. STATEMENT OF THE PROBLEM

Consider dynamic system

$$\overset{\circ}{x} = A(t)x + \mu_0 f(x,t), \quad t \in T = [t_*, t^*], \tag{1}$$

where $x = x(t)$ is an n- vector; μ_0 is small parameter, all functions $A(t) \in \mathbf{R}^{n \times n}$, $f(x,t) \in \mathbf{R}^n$, $x \in \mathbf{R}^n$, $t \in T$, are belonging to class $\mathbf{C}^p(p \geq 2)$.

Let the initial state $x(t_*) = z$ of the system (1) be unknown. A priori information about unknown initial state of the system (1) is as follows $z \in X_0 = \{z \in \mathbf{R}^n : d_* \leq z \leq d^*\}$. The set X_0 is called a priori distribution of the initial state of the system (1).

Assume that the observation under the system (1) a sensor system $y = c'x + \xi$, $y \in \mathbf{R}$, was carried out for defining more exactly the set X_0, where $\xi(t)$, $t \in T$, is some function belonging to $\mathbf{C}^p$ of the measuring errors satisfying the constraints $\xi_* \leq \xi(t) \leq \xi^*$, $t \in T$. Let $y(t)$, $t \in T$, be the sensor signal.

A set $X_* = X(y(\cdot))$ that consist of those and only those elements z from the set X_0, where $\xi(t)$, $t \in T$, is called a posteriori distribution of the initial state of the system (1).

We consider the following linear estimate of the set X_* used for the optimization of the dynamic systems:

$$\max h'z, \ z \in X_*, \ z \in X, \tag{2}$$

where $h \in R^n$ is the observation direction.

The problem (2) is called posteriori problem of the optimal observation of quasilinear systems.

Let $x(z,t)$, $t \in T$, be a solution of the system (1) with initial state $x(t^*) = z$. Let us call it $g(z,t) = c'x(z,t)$, $t \in T$; $b_*(t) = y(t) - \xi_*$, $b_*(t) = y(t) - \xi^*$, $t \in T$.

Then the problem (2) is equal to the following problem

$$\begin{aligned} & h'z \to \max, \\ & b_*(t) \leq g(z,t) \leq b^*(t), \quad t \in T, \\ & d_* \leq z \leq d. \end{aligned} \tag{3}$$

The problem (3) is the nonlinear semi-infinite programming problem (optimal observation problem).

The vector z^0 is called a solution (optimal plan) of the optimal observation problem (3) if z^0 satisfies the conditions

$$h'z^0 \to \max,$$
$$b_*(t) \le g(z^0, t) \le b^*(t), \quad t \in T,$$
$$d_* \le z^0 \le d.$$

The determination of the solution z^0 is a composite nonlinear extremal problem, the vector z^0 depends on the parameter μ_0 $\left(z^0 = z^0(\mu_0)\right)$. It is difficult to solve this problem.

Following the basic principle of asymptotic methods, we shall consider family systems

$$\overset{\circ}{x} = A(t)x + \mu f(x,t), \quad t \in T, \tag{4}$$

under the condition $\mu \to 0$.

Then the problem (3) can be written as follows:

$$h'z \to \max,$$
$$b_*(t) \le g(z,t,\mu) \le b^*(t), \quad t \in T, \tag{5}$$
$$d_* \le z \le d^*,$$

where $g(z,t,\mu) = c'x(z,t,\mu)$, $x(z,t,\mu)$, $t \in T$, - is a solution of the system (4).

Definition: The totality of the n-vectors $z^{s0} = \left(z_0^0, \ldots, z_s^0\right)$ is called s-optimal asimptotic solution (plan) of the problem (5) if the vector $z^{s0}(\mu) = \sum_{k=0}^{s} \mu^k z_k^0$, $\mu \ne 0$, satisfies the conditions:

1) the discrepency of the plan $z^{s0}(\mu)$ of its quality criterion from that of the optimal plan is of order $o(\mu^s)$, $\mu \to 0$;

2) $\max_{t \in T} \max\left\{ g\left(z^{s0}(\mu), t, \mu\right) - b^*(t), b_*(t) - g\left(z^{s0}(\mu), t, \mu\right), 0 \right\} = o(\mu^s)$, $\mu \to 0$.

In this paper we will describe an algorithm by means of which, for a given natural number $s \le p-1$, an s-optimal asimptotic solution can be constructed for the optimal observation problem.

2. THE BASIC PROBLEM

The first stage of the algorithm consists of solving the following problem which it is obtained from the problem (5) for $\mu = 0$:

$$h'z \to \max,$$
$$b_*(t) \le a'(t)z \le b^*(t), \quad t \in T, \tag{6}$$
$$d_* \le z \le d^*,$$

where $a'(t)z = c'F(t)z = g(z,t,0)$, $t \in T$; $F(t)$, $t \in T$, - is the solution of the matrix equation $\dot{F}(t) = A(t)F(t)$, $F(t_*) = E$.

The problem (6) is called the basic problem . Suppose that $\forall t \in T$ there is a vector $\nu \in \mathbf{R}^n$, $d_* \le \nu \le d^*$, such that

$$b^*(t) < a'(t)\nu < b^*(t), \quad \forall\, t \in T \quad \text{(Slater's condition)}.$$

Now we describe the solution [2] of the problem (6). Let z_0^0 be the optimal plan of the problem (6). Define

$$T_a = T_a^+ \cup T_a^- \,,\; T_a = \left\{t_{0i}^0,\; i \in I\right\},\; I = \left\{1,2,\ldots,l\right\},$$
$$T_a^+ = \left\{t_{0i}^0,\; i \in I^+\right\} = \left\{t \in T\colon g\left(z^0, t, 0\right) = b^*(t)\right\},\;\; I^+ = \left\{1,2,\ldots,l_1\right\},$$
$$T_a^- = \left\{t_{0i}^0,\; i \in I^-\right\} = \left\{t \in T\colon g\left(z^0, t, 0\right) = b^*(t)\right\},\;\; I^- = \left\{l_1 + 1,\ldots,l\right\},$$
$$I = I^+ \cup I^-;\;\; J_0^+ = \left\{j \in J\colon z_{0j}^0 = d_j^*\right\},\;\; J_0^- = \left\{j \in J\colon z_{0j}^0 = d_{*j}\right\},$$
$$J = \left\{1,2,\ldots,n\right\}.$$

According to [2] there exist such numbers $u_{0i}^0, i \in I, u_{0i}^0 \geq 0,\; i \in I^+, u_{0i}^0 \leq 0,\; i \in I^-$, that for z_0^0 and the estimates $\Delta_j = \sum_{i \in I} a_j\left(t_{0i}^0\right) u_{0i}^0 - c_j \,,\; j \in J$, the following statemens

$$\Delta_j \geq 0 \;\text{ if }\; j \in J_0^-;$$
$$\Delta_j \leq 0 \;\text{ if }\; j \in J_0^+;$$
$$\Delta_j = 0 \;\text{ if }\; j \in J_0 = J \setminus \left(J_0^+ \cup J_0^-\right) \;\text{ are fullfield.}$$

The vector $u_0^0 = \left(u_{0i}^0, i \in I\right)'$, we'll call the optimal dual plan of the problem (6). A pair $\left\{z_0^0, u_0^0\right\}$ we'll call the solution of the problem (6).

The solution of the problem (6) is called non-singular if it satisfies the following relations

$$rank\begin{bmatrix} a_j(t), t \in T_a;\; a_j(t), t \in T_a \setminus \left\{t_*, t^*\right\} \\ j \in J_0 \end{bmatrix} = \left|J_0\right|,$$
$$rank\begin{bmatrix} a_j(t), t \in T_a \\ j \in J_0 \end{bmatrix} = \left|T_a\right| = l,\;\; u_{0i}^0 \neq 0, i \in I, d_{*j} < z_{0j}^0 < d_j^*, j \in J_0.$$

Assumptions:

a) the solution $\left\{z_0^0, u_0^0\right\}$ of the problem (6) is non-sigular;

b) $\Delta_j > 0$ if $j \in J_0^-$; $\Delta_j < 0$ if $j \in J_0^+$;

c) $\ddot{a}'(t) z_0^0 - \ddot{b}^*(t) \neq 0,\;\; t \in T_a^+$; $\ddot{a}'(t) z_0^0 - \ddot{b}_*(t) \neq 0,\;\; t \in T_a^-$.

3. THE BASIC THEORY

After solving the basic problem, we form the matrix

$$R(v, \mu) = \begin{bmatrix} \bar{g}\left(\bar{z}, t_i, \mu\right) - b^*\left(t_i\right),\;\; i \in I^+ \\ \bar{g}\left(\bar{z}, t_i, \mu\right) - b_*\left(t_i\right),\;\; i \in I^- \\ \dfrac{\partial \bar{g}\left(\bar{z}, t_i, \mu\right)}{\partial t} - \dot{b}^*\left(t_i\right),\;\; i \in I^+ \\ \dfrac{\partial \bar{g}\left(\bar{z}, t_i, \mu\right)}{\partial t} - \dot{b}_*\left(t_i\right),\;\; i \in I^- \\ \sum_{i \in I} u_i \dfrac{\partial \bar{g}\left(\bar{z}, t_i, \mu\right)}{\partial \bar{z}_j} - c_j,\;\; j \in J_0 \end{bmatrix}, \qquad (7)$$

where $\bar{z} = \left(\bar{z}_j = z_j,\, j \in J_0\right)'$, $\bar{g}\left(\bar{z}, t, \mu\right) = g\left(\bar{z}, z_{0j}^0, j \in J_0^+ \cup J_0^-; t, \mu\right)$,

$v = \left(\bar{z}, t_1, \ldots, t_l, u\right)$, $u = \left(u_1, \ldots, u_l\right)'$.

Theorem. If conditions a)-c) are satisfied in the problem (5) for sufficiently small μ a solution exists $\left\{z^0(\mu), u^0(\mu)\right\}$, $z_j^0(\mu) = z_{0j}^0$, $j \in J_0^+ \cup J_0^-$, $z^0(\mu) \in C^{p-1}$, $z^0(0) = z_0^0$; $u^0(\mu) = \left(u_i^0(\mu), i \in I\right)'$, $u_i^0(0) = u_{0i}^0$, $i \in I$; $u^0(\mu) \in C^{p-1}$; $t_i^0(\mu) \in C^{p-1}$, $i \in I$, $t_i^0(0) = t_{0i}^0$, $i \in I$, and the vector $v^0(\mu) = \left(\bar{z}_j^0(\mu) = z_j^0(\mu), j \in J_0;\; t_i^0(\mu),\, u_i^0(\mu),\, i \in I\right)'$ is the solution of the following equation:

$$R(v, \mu) = 0 \tag{8}$$

Proof. It follows from the theorem about the differentiability of solutions of differential equations on the unitial data and parameter that the vector function $R(v, \mu)$ in the domaine $\|v - v_0\| \le \mu$, $|\mu| \le \mu_1$, (ε, μ_1 are sufficiently small numbers) belongs a class $\mathbf{C}^{p-1}$.

It's obvious that $R(v^0, 0) = 0$ and we can see that the Jacobi matrix J_0 has the following structure:

$$J_0 = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & F_1 & 0 \\ 0 & F_2 & A_1' \end{bmatrix},$$

where $A_1 = \left[\begin{matrix} a_j\left(t_{0i}^0\right),\; t \in J_0 \\ i \in I \end{matrix}\right]$, $A_2 = \left[\begin{matrix} \dot{a}_j\left(t_{0i}^0\right),\; j \in J_0 \\ i \in I \end{matrix}\right]$,

$F_1 = diag\left(\ddot{a}'\left(t_{0i}^0\right)z_0^0 - \ddot{b}^*\left(t_{0i}^0\right),\, i \in I^+;\; \ddot{a}'\left(t_{0i}^0\right)z_0^0 - \ddot{b}^*\left(t_{0i}^0\right),\, i \in I^-\right)$, $F_2 = A_2'\, diag\left(u_{0i}^0, i \in I\right)$, and it's non- degenerate according to the conditions a), c). Then the system (8) satisfies all the conditions of the implicit function theorem. According to this theorem, in some heighbourhood of zero $|\mu| \le \mu_1$ there are uniquely defined vector-function $v^0(\mu) \in \mathbf{C}^{p-1}$ satisfying system (8) such that $v^0(0) = v^0$. If follows from $z^0(\mu) \to z_0^0$ for $\mu \to 0$ that $\left|T_a(\mu)\right| = \left|\left\{t_i^0(\mu), i = 1, 2, \ldots, \bar{l}\right\}\right| \le \left|T_a\right|$.

Let $\left|T_a(\mu)\right| = \left|T_a\right|$ $\left(\bar{l} = l\right)$. Assume $z^0(\mu)$ are non-optimal plan of the problem (5). Then there exists a an optimal plan $z^*(\mu)$ and $z^*(\mu) \to z^*$ for $\mu \to 0$. It follows $c'z^*(\mu) > c'z^0(\mu)$ than $c'z^* > c'z_0^0$. We have a contrudicthion agaist the assumption .

Now let $\left|T_a(\mu)\right| = \bar{l} \ne l$. Then for the plan $z^0(\mu)$ there exists the numbers $u_i(\mu), t_i(\mu)$, $i = 1, 2, \ldots, \bar{l}$ satisfying (8). But $\left|u_i(\mu)\right| \le M$, $M > 0$, $t_* < t_i(\mu) < t^*$, $i = 1, 2, \ldots, \bar{l}$, then there exists the sequence $\left\{\mu_k\right\}_{k=\overline{0,\infty}}$, $\mu_k \to 0$ for $k \to \infty$ such that $u_i(\mu_k) \to u_i(0) = u_i$; $t_i(\mu_k) \to t_i(0) = t_i$, $i = 1, 2, \ldots, \bar{l}$ for $k \to \infty$. Thus we obtain than the basic problem has two unequal optimal plans $u = \left(u_i, i = 1, \ldots, \bar{l}\right)'$, $u_0^0 = \left(u_{0i}^0, i \in I\right)'$ than it impossible. Therefore the vector $z^0(\mu)$ is the optimal plan. The theorem is proved.

4. CONSTRUCTIONS OF THE ASYMPTOTIC FORM

We now describe the algorithm to constructing an an s - optimal asymptotic plan of the problem (5). To construct an s-optimal asymptotic plan shall choose natural number s and find the Taylor's polynomials of the order s for the functions $z_j(\mu),\ j\in J,\quad u_i(\mu),\quad t_i(\mu), i\in I$:

$$z_j^s(\mu)=\sum_{k=0}^{s}\mu^k z_{kj},\quad j\in J,$$

$$u_i^s(\mu)=\sum_{k=0}^{s}\mu^k u_{ki},\ t_i^s(\mu)=\sum_{k=0}^{s}\mu^k t_{ki},\ i\in I.$$

Let

$$g(z,t,\mu)=c'x(z,t,\mu)=\sum_{k=0}^{s}\mu^k g_k(z,t)+O_1(\mu^s),\tag{9}$$

$$g_k(z,t)=c'x_k(z,t),\quad k=0,1,\dots,s.$$

The differential equations for the functions $x_k(z,t),\quad k=0,1,\dots,s$, can be obtained by applying the Poincare formalism to system (4):

$$\dot{x}_0(z,t)=Ax_0(z,t),\ x_0(z,0)=z,$$

$$\dot{x}_1(z,t)=Ax_1(z,t)+f(x_0(z,t),t),\ x_1(z,0)=0,$$

$$\dot{x}_2(z,t)=Ax_2(z,t)+\frac{\partial f'(x_0(z,t),t)}{\partial x}x_1(z,t),\ x_2(z,0)=0,\qquad t\in T,$$

$$\dots$$

The vector function $R(\nu,\mu)$ defined in the domain $\|\nu-\nu^0\|\le\varepsilon,\quad |\mu|\le\mu_1$, where ε and μ_1 are fairly positive numbers, can be expanded in form:

$$R(\nu,\mu)=\sum_{k=0}^{s}\mu^k R_k(\nu)+O_2(\mu_s),$$

where

$$R_0(\nu)=\begin{bmatrix} g_0(z,t_i)-b^*(t_i),\ i\in I^+\\ g_0(z,t_i)-b_*(t_i),\ i\in I^-\\ \dfrac{\partial g_0(z,t_i)}{\partial t}-\dot{b}^*(t_i),\ i\in I^+\\ \dfrac{\partial g_0(z,t_i)}{\partial t}-\dot{b}_*(t_i),\ i\in I^-\\ \sum_{i\in I}u_i\dfrac{\partial g_0(z,t_i)}{\partial z_j}-c_j,\ j\in J_0\end{bmatrix},\quad R_k(\nu)=\begin{bmatrix} g_k(z,t_i),\ i\in I^+\\ g_k(z,t_i),\ i\in I^-\\ \dfrac{\partial g_k(z,t_i)}{\partial t},\ i\in I^+\\ \dfrac{\partial g_k(z,t_i)}{\partial t},\ i\in I^-\\ \sum_{i\in I}u_i\dfrac{\partial g_k(z,t_i)}{\partial z_j},\ j\in J_0\end{bmatrix},\quad k\ge 1.$$

Now we form the systems of linear quations for coefficients $z_{kj}^0,\quad j\in J_0;\quad u_{ki}^0,\ t_{ki}^0,\ i\in I$. Using Taylor's formula , we expend the function $R(\nu^s,\mu)=\sum_{k=0}^{s}\mu^k R_k(\nu^s(\mu))$ ($\nu^s(\mu)=\sum_{k=0}^{s}\mu^k\nu_k$) in powers of μ to including order **s** and equate coefficients of the expension to zero. As a result, we will obtain non - degenerate systems of linear equations from which to find in succession the vectors $\nu_k^0,\quad k=0,1,\dots,s$:

$$J_0 v_1^0 = -R_1(v_0^0),$$

$$J_0 v_2^0 = -\frac{\partial R_1(v_0^0)}{\partial v} v_1^0 - \frac{1}{2} v_1^{0\prime} \frac{\partial^2 R_0(v_0^0)}{\partial v^2} v_1^0 - R_2(v_0^0),$$

$$\ldots$$

Notice that, the matrix J_0 is Jacobi matrix of the equation (8). By successively solving system (10), we find coefficients z_{kj}^0, $j \in J_0$, u_{ki}^0, t_{ki}^0, $i \in I$, $k = 0, 1, \ldots, s$ and from the polinomials $z_j^{s0}(\mu)$, $j \in J_0$; $u_i^{s0}(\mu)$, $t_i^{s0}(\mu)$, $i \in I$. The vector ($z_j^{s0}(\mu)$, $j \in J$; z_j^0, $j \in J_0^+ \cup J_0^-$) well be s-optimal asymptotic plan of the problem (5).

The asymptotic approximations of the roots of the system (8) thus constructed can be used for the numerical solution of this sistem and therefore also for the problem with a prescribed value of the small parameter. To be able to do this, we need to apply the final adjustment procedure [3], that is, to use Newton's method to find the roots of the system (8).

5. EXAMPLE

Consider the dynamic system of the form:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\mu x_2^2,\ x_1(0) = z_1,\ x_2(0) = z_2,\ t \in T = [0, 4],$$

$$z \in X_0 = \{z \in \mathbf{R}^2 \mid 0 \le z_1 \le 2,\ 0 \le z_2 \le 3\}.$$

Let $c = (1, 1)'$, $\xi(t) = \sin 5t$, $t \in T$, $\xi_* = -1$, $\xi^* = 1$, $h = (0, 1)'$. The vector $y(t)$, $t \in T$, for $\mu_0 = 0{,}01$ is equal to $1 + 100 \ln(0{,}01t + 1) + \frac{1}{0{,}01t + 1} + \sin 5t$, $t \in T$.

Then the problem (5) is described in the following way:

$$z_2 \to \max,\quad 100\ln(0{,}01t + 1) + \frac{1}{0{,}01t + 1} + \sin 5t \le$$

$$\le z_1 + \frac{1}{\mu}\ln(\mu z_2 t + 1) + \frac{z_2}{(\mu z_2 t + 1)} \le$$

$$\le 100\ln(0{,}01t + 1) + \frac{1}{0{,}01t + 1} + \sin 5t + 2,\ t \in T = [0, 4],$$

$$0 \le z_1 \le 2,\ 0 \le z_2 \le 3.$$

The basic problem in this case has the form

$$z_2 \to \max,\quad \frac{1}{0{,}01t + 1} + 100\ln(0{,}01t + 1) + \sin 5t \le$$

$$\le z_1 + z_2 + t z_2 \le \frac{1}{0{,}01t + 1} + 100\ln(0{,}01t + 1) + \sin 5t + 2,\ t \in T,$$

$$0 \le z_1 \le 2,\ 0 \le z_2 \le 3.$$

From here we obtain the following solution

$$z_{01}^0 = 1{,}059753,\ z_{02}^0 = 0{,}965993\ ,$$

$$u_{01}^0 = 0{,}530518,\ u_{02}^0 = -0{,}530518,$$

$$t_{01}^0 = 3{,}456102,\ t_{02}^0 = 1{,}571150.$$

We will construct the 1-optimal asymptotic plan of the observation problem. Find the vector v_1^0 from (10):

$$J_0 v_0^1 = -R_1(v_0^0),$$

where

$$J_0 = \begin{bmatrix} 1 & 1+t_{01}^0 & 0 & 0 & 0 & 0 \\ 1 & 1+t_{02}^0 & 0 & 0 & 0 & 0 \\ 0 & 1 & f_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & f_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & u_{01}^0 & u_{02}^0 & 1+t_{01}^0 & 1+t_{02}^0 \end{bmatrix}, \quad -R_1(v_0^0) = \begin{bmatrix} z_{02}^{0\,2}\left(t_{01}^0 + \frac{t_{01}^{0\,2}}{2}\right) \\ z_{02}^{0\,2}\left(t_{02}^0 + \frac{t_{02}^{0\,2}}{2}\right) \\ z_{02}^{0\,2}\left(1+t_{01}^0\right) \\ z_{02}^{0\,2}\left(1+t_{02}^0\right) \\ u_{01}^0 z_{02}^0\left(2t_{01}^0 + t_{01}^{0\,2}\right) + u_{02}^0 z_{02}^0\left(2t_{02}^0 + t_{02}^{0\,2}\right) \end{bmatrix}.$$

Now we define v_1^0 that is

$$z_{11}^0 = -5{,}8122210\ ,\ z_{12}^0 = 3{,}2787137,$$
$$u_{11}^0 = 3{,}6013101,\ u_{12}^0 = -3{,}6013101,$$
$$t_{11}^0 = -0{,}035191527,\ t_{12}^0 = -0{,}3516526.$$

Thus 1-optimal asymptotic plan of the observation problem is equal to

$$z_1^{10}(\mu) = 1{,}059753 + \mu(-5{,}812221),$$
$$z_2^{10}(\mu) = 0{,}965993 + \mu(3{,}2787137).$$

For $\mu = 0{,}01$, $z_1^{10}(0{,}01) = 1{,}001631$, $z_2^{10}(0{,}01) = 0{,}998780137$.

Note that the correct initial state of the system ($y(t)$, $t \in T$, $\mu_0 = 0{,}01$) is $z_1 = 1$, $z_2 = 1$.

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