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General Description of Dirac Particle in Riemannian Space–Times

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We discuss dynamics of Dirac fermions in strong gravitational fields. The general Foldy–Wouthuysen Hamiltonian is derived. Particle dynamics in strong fields is considered.

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The study of spin dynamics in a curved space–time (a gravitational field) was initiated immediately after the formulation of the relativistic Dirac theory. The early efforts were mainly concerned with the development of mathematical tools and methods appropriate for the description of interaction of spinning particles with a gravitational field. The studies of the spinor analysis in the framework of the general Lagrange-Noether approach have subsequently resulted in the construction of the gauge-theoretic models of physical interactions, including also gravity (see Ref. [1] and references therein).

We present the results of our investigations of the Dirac fermions based on the new method [2] of the Foldy–Wouthuysen (FW) transformation. Earlier, we analyzed the dynamics of spin in weak static and stationary gravitational fields [3, 4] and in strong stationary gravitational fields [5] of massive compact sources. These previous results are now extended to the general case of a completely arbitrary gravitational field.

Our notations and conventions are the same as in Ref. [5].

We use the notations t and x^a ($a = 1,2,3$) for the coordinate time and the spatial local coordinates, respectively. A convenient parametrization of the space–time metric was proposed by De Witt [6] in the context of the canonical formulation of the quantum gravity theory. In a slightly different disguise, the general form of the line element of an arbitrary gravitational field reads

$$\begin{aligned} ds^2 = V^2c^2dt^2 - \delta_{\widehat{a}\widehat{b}}W^{\widehat{a}}{}_cW^{\widehat{b}}{}_d \\ \times (dx^c - K^c cdt)\,(dx^d - K^d cdt). \end{aligned} \tag{1}$$

One needs orthonormal frames to discuss the spinor field and to formulate the Dirac equation. From the mathematical point of view, the tetrad is necessary to "attach" a spinor space at every point of the space–time manifold. Tetrads (coframes) are naturally defined up to a local Lorentz transformations, and one usually fixes this freedom by choosing a gauge. We discussed the choice of the tetrad gauge in [4] and have demonstrated that a physically preferable option is the Schwinger gauge [7, 8], namely the condition $e_a^{\widehat{0}} = 0$. Accordingly, for the general metric (1) we will work with the tetrad

$$\begin{aligned} &e_i^{\widehat{0}} = V\,\delta_i^0, \quad e_i^{\widehat{a}} = W^{\widehat{a}}{}_b\left(\delta_i^b - cK^b\,\delta_i^0\right), \\ &a = 1,2,3. \end{aligned} \tag{2}$$

The inverse tetrad, such that $e_\alpha^i e_j^\alpha = \delta_j^i$,

$$\begin{aligned} &e_{\widehat{0}}^i = \frac{1}{V}\left(\delta^i_0 + \delta^i_a cK^a\right), \quad e_{\widehat{a}}^i = \delta^i_b W^b{}_{\widehat{a}}, \\ &a = 1,2,3, \end{aligned} \tag{3}$$

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also satisfies the similar Schwinger condition, $e^0_{\widehat{a}} = 0$. Here we introduced the inverse 3×3 matrix, $W^a{}_{\widehat{c}} W^{\widehat{c}}{}_b = \delta^a_b$.

The Dirac equation in a curved space–time reads

$$(i\hbar\gamma^\alpha D_\alpha - mc)\Psi = 0, \qquad \alpha = 0, 1, 2, 3. \tag{4}$$

The Dirac matrices γ^α are defined in local Lorentz (tetrad) frames. They have constant components. The spinor covariant derivatives are [1, 9, 10]

$$D_\alpha = e^i_\alpha D_i, \qquad D_i = \partial_i + \frac{iq}{\hbar} A_i + \frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\,\alpha\beta}. \tag{5}$$

Here the Lorentz connection is $\Gamma_i{}^{\alpha\beta} = -\Gamma_i{}^{\beta\alpha}$, and $\sigma^{\alpha\beta} = \frac{i}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)$ are the generators of the local Lorentz transformations of the spinor field. We introduced the useful objects:

$$\begin{aligned}
&\mathcal{F}^a{}_b = VW^a{}_{\widehat{b}},\\
&\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}} W^d{}_{\widehat{b}}\left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e\partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e\partial_d K^e\right),\\
&\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}} W^e{}_{\widehat{b}}\,\partial_{[d} W^{\widehat{c}}{}_{e]},\\
&\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}}\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}.
\end{aligned} \tag{6}$$

We also introduced a pseudoscalar Υ and a 3-vector $\mathbf{\Xi} = \{\Xi_a\}$ by [5]

$$\begin{aligned}
&\Upsilon = V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}},\\
&\Xi_{\widehat{a}} = \frac{V}{c}\,\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{0}}{}^{\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\mathcal{Q}^{\widehat{b}\widehat{c}}
\end{aligned} \tag{7}$$

where $\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = -\mathcal{C}_{\widehat{b}\widehat{a}}{}^{\widehat{c}}$ is the anholonomity object for the spatial triad $W^{\widehat{a}}{}_b$.

We limit ourselves to the case when an electromagnetic field is switched off. After a lengthy algebra, we obtain the FW Hamiltonian in the form

$$\begin{aligned}
\mathcal{H}_{FW} &= \mathcal{H}^{(1)}_{FW} + \mathcal{H}^{(2)}_{FW}, \quad \mathcal{H}^{(1)}_{FW} = \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, \left(2\epsilon^{cae}\Pi_e\{p_b, \mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\}\right.\right.\\
&\left.\left. + \Pi^a\{p_b, \mathcal{F}^b{}_a\Upsilon\}\right)\right\} + \frac{\hbar m c^4}{4}\epsilon^{cae}\Pi_e\left\{\frac{1}{\mathcal{T}}, \left\{p_d, \mathcal{F}^d{}_c\mathcal{F}^b{}_a\partial_b V\right\}\right\},\\
\mathcal{H}^{(2)}_{FW} &= \frac{c}{2}\left(K^a p_a + p_a K^a\right) + \frac{\hbar c}{4}\Sigma_a\Xi^a + \frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_a\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \left[\epsilon^{abc}\left(\frac{1}{c}\dot{\mathcal{F}}^f{}_c\right.\right.\right.\right.\right.\\
&\left.\left.\left.\left.\left. - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c\right) - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\},\\
\epsilon' &= \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{p_b, \mathcal{F}^b{}_a\}\{p_d, \mathcal{F}^d{}_c\}}, \quad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\}.
\end{aligned} \tag{8}$$

The equation of spin motion is obtained from the commutator of the FW Hamiltonian with the polarization operator $\mathbf{\Pi} = \beta\mathbf{\Sigma}$ [11]:

$$\begin{aligned}
&\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi}\ ,\\
&\Omega^a_{(1)} = \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \{p_e, \epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d V\}\right\} + \frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \{p_e, (2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\Upsilon)\}\right\},
\end{aligned} \tag{9}$$

$$\Omega^a_{(2)} = \frac{\hbar c^2}{8}$$

$$\times \left\{ \frac{1}{\mathcal{T}}, \left\{ \{p_e, \mathcal{F}^e{}_b\}, \left\{ p_f, \left[\epsilon^{abc} (\frac{1}{c} \dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c \partial_d K^f + K^d \partial_d \mathcal{F}^f{}_c) - \frac{1}{2} \mathcal{F}^f{}_d \left(\delta^{db} \Xi^a - \delta^{da} \Xi^b \right) \right] \right\} \right\} \right\} + \frac{c}{2} \Xi^a.$$

The explicit expression for the force operator reads [11]

$$\begin{aligned} F_{\widehat{a}} &= \frac{1}{2} \left\{ \dot{W}^b{}_{\widehat{a}}, p_b \right\} + \frac{1}{4} \left\{ p_b, \left\{ \frac{\partial \mathcal{H}_{FW}}{\partial p_c}, \partial_c W^b{}_{\widehat{a}} \right\} \right\} - \frac{1}{2} \left\{ W^b{}_{\widehat{a}}, \partial_b \mathcal{H}_{FW} \right\}, \\ \frac{\partial \mathcal{H}_{FW}}{\partial p_c} &= \beta \frac{c^2}{4} \delta^{ad} \left\{ \frac{1}{\epsilon'}, \left\{ p_b, \mathcal{F}^b{}_a \mathcal{F}^c{}_d \right\} \right\} + cK^c + \frac{\hbar}{2} \mathfrak{T}^c \end{aligned} \quad (10)$$

where we introduced the following compact notation

$$\mathfrak{T}^c = \frac{\partial \mathcal{U}}{\partial p_c}, \qquad \mathcal{U} := \mathbf{\Pi} \cdot \mathbf{\Omega}_{(1)} + \mathbf{\Sigma} \cdot \mathbf{\Omega}_{(2)}. \quad (11)$$

Thus, we start from the covariant Dirac equation, apply the FW transformation [2], and construct the FW Hamiltonian for an *arbitrary space–time geometry.* We also derive the operator equations of motion.

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