ON CIRCULAR DISARRANGED STRINGS OF SEQUENCES

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Two sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$, sharing $n-1$ elements, are said disarranged if for every subset $Q \subseteq [n]$, the sets $\{a_i \mid i \in Q\}$ and $\{b_i \mid i \in Q\}$ are different. In this paper we investigate properties of these pairs of sequences. Moreover we extend the definition of disarranged pairs to a circular string of n-sequences and prove that, for every positive integer m, except some initials values for n even, there exists a similar structure of length m.

Introduction

Let n be a positive integer, $R = (a_1, a_2, \dots, a_n)$ and $S = (b_1, b_2, \dots, b_n)$ n-sequences of distinct elements, sharing exactly $n-1$ elements.

We associate with R and S the bijection f defined by the relation $f(a_i) = b_i$, $1 \leq i \leq n$, and represented in two line notation by the $2 \times n$ array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}.$$

Let u and v be the different elements which belong to the first and the second line respectively. The function f is formed by the linear ordering $l(f) = (u, f(u), f^2(u), \dots, f^{k-1}(u), v)$, where k is the minimum positive integer such that $f^k(u) = v$, and a permutation $\pi(f)$ on the remaining elements. In [2] a similar function, called widened permutation, is investigated in the context of the theory of species of Joyal. We say that R and S are **disarranged** if for every set
$\{i_1, i_2, \dots, i_r\} \subseteq \{1, 2, \dots, n\}$ $\{a_{i_1}, a_{i_2}, \dots, a_{i_r}\} \neq \{b_{i_1}, b_{i_2}, \dots, b_{i_r}\}$.
The sequences R and S are called **1-disarranged** if there exists an index $i \in [n]$ such that $a_i = b_i$ and the sequences, obtained from R and S after deleting a_i and b_i, are disarranged. In this case we say that the pair (R, S) is 1-disarranged.

We extend the definition to a string of n-sequences.

Definition 1. Let $n, m \in \mathbb{N}$; an m-string $(S_1, S_2, \dots, S_m)$ of n-sequences, is called **disarranged** if:

(A1) S_i is disjoint from S_{i-1} and S_{i+1},

(A2) S_{i-1} and S_{i+1} are disarranged.

for every $i = 2, \dots, m-1$.

A disarranged m-string of n-sequences is *circular* when the properties **(A1)** and **(A2)** are satisfied for every $i = 1, 2, \dots, m$ (taking the indices modulo m).

Main results

The notion of circular disarranged string of n-sequences has application in relation to an edge coloring problem of graphs [4]. In this paper we investigate properties of disarranged pairs of sequences and circular disarranged string of n sequences. In particular we prove that the n-sequences R and S, sharing exactly $n-1$ elements are disarranged if and only if the linear ordering

$l(R, S)$ contains all the elements of R and S. Moreover we prove that, for every positive integer m, there exists a circular disarranged string of n sequences of length m, except some initials values for n even.

The following theorem is a consequence of some Lemmas and Propositions.

Theorem 1. *Let m, n be positive integers. For n odd and every $m > 2$ or for n even and $m > 6$ even ($m \neq 14$) or for $m \geq 2n + 1$ odd ($m \neq 2n + 7$), there exists a circular disarranged m-string. For the remaining cases, there exists a circular 1-disarranged m-string.*

References

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