ON CIRCULAR DISARRANGED STRINGS OF SEQUENCES

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Two sequences \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\), sharing \(n - 1\) elements, are said disarranged if for every subset \(Q \subseteq [n]\), the sets \(\{a_i \mid i \in Q\}\) and \(\{b_i \mid i \in Q\}\) are different. In this paper we investigate properties of these pairs of sequences. Moreover we extend the definition of disarranged pairs to a circular string of \(n\)-sequences and prove that, for every positive integer \(m\), except some initials values for \(n\) even, there exists a similar structure of length \(m\).

Introduction

Let \(n\) be a positive integer, \(R = (a_1, a_2, \ldots, a_n)\) and \(S = (b_1, b_2, \ldots, b_n)\) \(n\)-sequences of distinct elements, sharing exactly \(n - 1\) elements.

We associate with \(R\) and \(S\) the bijection \(f\) defined by the relation \(f(a_i) = b_i, 1 \leq i \leq n\), and represented in two line notation by the \(2 \times n\) array

\[
\begin{pmatrix}
  a_1 & a_2 & \ldots & a_n \\
  b_1 & b_2 & \ldots & b_n
\end{pmatrix}
\]

Let \(u\) and \(v\) be the different elements which belong to the first and the second line respectively. The function \(f\) is formed by the linear ordering \(l(f) = (u, f(u), f^2(u), \ldots, f^{k-1}(u), v)\), where \(k\) is the minimum positive integer such that \(f^k(u) = v\), and a permutation \(\pi(f)\) on the remaining elements. In [2] a similar function, called widened permutation, is investigated in the context of the theory of species of Joyal. We say that \(R\) and \(S\) are disarranged if for every set \(\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}\) \(\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \neq \{b_{i_1}, b_{i_2}, \ldots, b_{i_r}\}\).

The sequences \(R\) and \(S\) are called 1-disarranged if there exists an index \(i \in [n]\) such that \(a_i = b_i\) and the sequences, obtained from \(R\) and \(S\) after deleting \(a_i\) and \(b_i\), are disarranged. In this case we say that the pair \((R, S)\) is 1-disarranged.

We extend the definition to a string of \(n\)-sequences.

Definition 1. Let \(n, m \in \mathbb{N}\); an \(m\)-string \((S_1, S_2, \ldots, S_m)\) of \(n\)-sequences, is called disarranged if:

(A1) \(S_i\) is disjoint from \(S_{i-1}\) and \(S_{i+1}\),

(A2) \(S_{i-1}\) and \(S_{i+1}\) are disarranged.

for every \(i = 2, \ldots, m - 1\).

A disarranged \(m\)-string of \(n\)-sequences is circular when the properties (A1) and (A2) are satisfied for every \(i = 1, 2, \ldots, m\) (taking the indices modulo \(m\)).

Main results

The notion of circular disarranged string of \(n\)-sequences has application in relation to an edge coloring problem of graphs [4]. In this paper we investigate properties of disarranged pairs of sequences and circular disarranged string of \(n\) sequences. In particular we prove that the \(n\)-sequences \(R\) and \(S\), sharing exactly \(n - 1\) elements are disarranged if and only if the linear ordering
$l(R, S)$ contains all the elements of $R$ and $S$. Moreover we prove that, for every positive integer $m$, there exists a circular disarranged string of $n$ sequences of length $m$, except some initials values for $n$ even.

The following theorem is a consequence of some Lemmas and Propositions.

**Theorem 1.** Let $m, n$ be positive integers. For $n$ odd and every $m > 2$ or for $n$ even and $m > 6$ even ($m \neq 14$) or for $m \geq 2n + 1$ odd ($m \neq 2n + 7$), there exists a circular disarranged $m$-string. For the remaining cases, there exists a circular 1-disarranged $m$-string.

**References**


