ON INTERSECTION OF TRIPLE OF PREFRATTINI SUBGROUPS IN FINITE SOLUBLE GROUP

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In this paper we consider finite groups only, so the term "group" always means "finite group".

D.S. Passman in [1] proved that a p-soluble group G always possesses three Sylow p-subgroups such that their intersection is equal to $O_p(G)$. Later V.I. Zenkov [2] proved the same statement for an arbitrary group. In [3–4] S. Dolfi proved that in every π-soluble group G there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = O_\pi(G)$ holds.

In connection with these results, in the *Kourovka Notebook* [5] the author formulated the following Problem 17.55:

Does there exist an absolute constant k such that for any prefrattini subgroup H in any finite soluble group G there exist k conjugates of H whose intersection is $\Phi(G)$, the Frattini subgroup of G?

The main goal of this paper is to give an affirmative answer to this question.

Theorem. *Let H be a prefrattini subgroup of a soluble group G. Then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = \Phi(G)$ holds.*

Corollary 1. *Let H be a prefrattini subgroup of a soluble group G. If $\Phi(G) = 1$, then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = 1$ holds.*

Corollary 2. *Let H be a prefrattini subgroup of a soluble group G. Then the inequalities $|H| \leq \sqrt[3]{|G|^2 \cdot |\Phi(G)|}$ and $|H/\Phi(G)| \leq |G : H|^2$ hold.*

Corollary 3. *Let H be a prefrattini subgroup of a soluble group G. If $\Phi(G) = 1$, then the inequalities $|H| \leq \sqrt[3]{|G|^2}$ and $|H| \leq |G : H|^2$ hold.*

The concept of a prefrattini subgroup of a soluble group was introduced by Gaschütz in [6]. Considering a complemented chief factor L/K of a soluble group G as a G-module, Gaschütz proved that G has a normal section that is a completely reducible G-module whose composition components are G-isomorphic to L/K, and its composition length m is equal to the number of complemented and G-isomorphic to L/K factors of a chief series of G. That section is denoted by $Cr_G(L/K)$ and called a *crown of G corresponding to L/K*. A constructive definition of a crown of a soluble group G corresponding to a complemented chief factor L/K is given as follows:

$$Cr_G(L/K) = C_G(L/K)/R,$$

where R is the intersection of the cores of maximal subgroups complementing L/K.

A crown of a soluble group G is a crown corresponding to a complemented chief factor of G. The set of all crowns of G is denoted by $Cr(G)$. From the Jordan-Hölder theorem it follows that for the construction of $Cr(G)$ it is enough to consider some chief series of G and to choose in it the maximal system $L_1/K_1, \dots, L_t/K_t$ pairwise non-G-isomorphic complemented chief factors. Then we have $Cr(G) = \{Cr_G(L_1/K_1), ..., Cr_G(L_t/K_t)\}$.

Definition. Let G be a soluble group, and $Cr(G) = \{Cr_G(L_1/K_1), ..., Cr_G(L_t/K_t)\}$. Let G_i be a complement of $Cr_G(L_i/K_i)$ in G, where $i \in I = \{1, 2, ..., t\}$. Then the subgroup $\bigcap_{i \in I} G_i$ is called a prefrattini subgroup of G.

By Definition, every soluble group has at least one prefrattini subgroup.

The following theorem gives basic properties of prefrattini subgroups.

Theorem ([6]). *For a soluble group G, the following conditions hold*:

1) *if H is a prefrattini subgroup of G and $N \lhd G$, then*:

a) *H^x is a prefrattini subgroup of G for any element $x \in G$*;

b) *HN/N is a prefrattini subgroup of G/N*;

c) *H covers all Frattini chief factors of G and avoids all complemented chief factors of G*;

d) $Core_G(H) = \Phi(G)$;

e) *$|H|$ is the product of the orders of all Frattini chief factors in a chief series of G*;

2) *any two prefrattini subgroups of G are conjugate in G.*

References

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