SPECTRAL DECOMPOSITION OF AN INCIDENCE STRUCTURE

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Let X and Y be finite sets and let S be a subset of $X \times Y$. Then $\mathcal{S} = (X, Y; S)$ is an *incidence structure* on $(X,\, Y)$, and we say that $x \in X$ is *incident with* $y \in Y$ if (x, y) belongs to S. Many combinatorial objects can be described efficiently in the language of incidence structures. These includes graphs, codes, designs, simplicial complexes, ranked partially ordered sets, and so on. The automorphism group $G := Aut(\mathcal{S})$ of $\mathcal{S}$ consists of all pairs $(g, h) \in \mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ so that $(x, y) \in S$ if and only if (x^g, y^h). In particular, we have two distinct actions of G, one on X and one on Y.

In this note we discuss the relationship between these two actions at the level of permutation modules for X and Y.

Let $R = \mathbb{C}$ and denote the free R-module with basis X by RX. The elements of RX thus are all formal sums $\sum_{x \in X} r_x\, x$ where $r_x \in R$. On RX we have the standard inner product $(\ ,\ )$ by setting $(x, x') = 0$ if $x \neq x' \in X$ and $(x, x) = 1$ otherwise. Evidently RX is a permutation module for G by setting $(\sum_{x \in X} r_x\, x)^g := \sum_{x \in X} r_x\, x^g$ for $g \in G$.

To the incidence structure $\mathcal{S} = (X, Y; S)$ now associate two linear *incidence maps* in a standard fashion:

$$\varepsilon\colon\ RX \to RY \quad \text{and} \quad \partial\colon\ RY \to RX,$$

defined on the respective bases by

$$\varepsilon(x) = \sum_{y\colon (x,y) \in S} y \quad \text{for } x \in X \quad \text{and} \quad \partial(y) = \sum_{x\colon (x,y) \in S} x \quad \text{for } y \in Y\,.$$

These maps are adjoints of each other and therefore the maps

$$\nu^+ = \partial\varepsilon\colon\ \ RX \to RX \quad \text{and} \quad \nu^- = \varepsilon\partial\colon\ \ RY \to RY$$

are symmetric with respect to the inner products on RX and RY. Evidently all eigenvalues are real and non-negative.

A simple argument shows that any non-zero eigenvalue of ν^+ is also an eigenvalue of ν^- and vice versa. Furthermore, for such a non-zero eigenvalue the two corresponding eigenspaces are isomorphic to each other via the restriction of ε or ∂. Therefore it make sense to speak of the *non-zero eigenvalues* of $\mathcal{S} = (X, Y; S)$.

Theorem (Spectral Decomposition) *Let $\mathcal{S} = (X, Y; S)$ be a finite incidence structure with eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_t > 0$. Denote the corresponding eigenspaces by $E_0,\, E_1, .., E_t \subseteq RX$ and $E'_0,\, E'_1, .., E'_t \subseteq RY$. Further, denote the kernel of ε and ∂ by $K_X \subseteq RX$ and $K_Y \subseteq RY$, respectively, and let G be the automorphism group of $\mathcal{S}$. Then*

$$\begin{aligned} RX &= E_0 \oplus E_1 \oplus \ldots \oplus E_t \oplus K_X \quad \textit{and} \\ RY &= E'_0 \oplus E'_1 \oplus \ldots \oplus E'_t \oplus K_Y \end{aligned} \tag{1}$$

are G-invariant orthogonal decompositions into pairwise isomorphic G-modules $E_i \simeq E'_i$ for all $i = 0, .., t$.

We note that the actual decomposition is obtained in a standard fashion which is completely explicit. The projection $\pi_i : RX \to E_i$ is a polynomial expression in the λ_0, λ_1, ..., λ_t and ν^+. Therefore any element in f in RX is decomposed as $f = f_0 + f_1 + ... + f_t + f_X$ with $f_i = \pi_i(f) \in E_i$ and $f_X \in K_X$. Such explicit formulae are essential in many computations.

For regular graphs (take X=vertices and Y=edges of the graph) this decomposition coincides with the decomposition afforded by the usual graph spectrum. For association schemes (take $X = Y$ and define 'incidence' by 'i-association' for a suitable i) the eigenspaces are closely related (identical in most cases) to the decomposition afforded by the minimal idempotents of the scheme. Fundamental properties of association schemes, such as Delsarte's Linear Programming Bound and estimates for inner and outer distributions are immediate from the explicit decomposition just mentioned. But we note that these properties of association schemes transfer naturally to incidence structures in general.

For a finite projective space $\mathrm{PG}(n, q)$ we may consider the incidence structure of s-versus t-dimensional subspaces of $\mathrm{PG}(n, q)$. Here the spectra are easy to compute, all spectral values turn out to be integral and all eigenspaces are irreducible $\mathrm{GL}(n, q)$-modules. The same is true for the $q = 1$ analogue when $\mathcal{S}$ is the incidence structure of s-versus t-dimensional subsets of a n-set.

These comments show that spectral compositions provide a unifying principle that opens up new perspectives. In this talk I will concentrate on orbits of automorphism groups of incidence structures. This is joint work [1,2] with Ben Summer at UEA and Francesca Dalla Volta at Milan Bicocca.

References

1. Dalla Volta F., Siemons, J. *On the spectral decomposition of an incidence structure* // to appear.
2. Siemons J., Summer B. *On the face complex of the hyperoctahedron* // to appear.