ON STRONGLY SUPERSOLUBLE FINITE GROUPS

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Throughout this report, all groups are finite. The notion of a normal subgroup takes a central place in the theory of groups. One of its generalizations is the notion of a modular subgroup, i.e. a modular element (in the sense of Kurosh [1, Chapter 2, p. 43]) of a lattice of all subgroups of a group. Recall that a subgroup $M$ of a group $G$ is called modular in $G$, if the following assertions hold:

1) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and
2) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

Properties of modular subgroups were studied in the book [1]. Groups with all subgroups are modular were studied by R. Schmidt [1], [2] and I. Zimmermann [3]. By parity of reasoning with subnormal subgroup, in [3] the notion of a submodular subgroup was introduced.

Definition 1 [3]. A subgroup $H$ of a group $G$ is called submodular in $G$, if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \ldots \leq H_s = G$ such that $H_i$ is a modular subgroup in $H_i$ for $i = 1, \ldots, s$.

Using this notion we introduce a key notion of this report.

Definition 2. A group $G$ we will call strongly supersoluble if $G$ is supersoluble and every Sylow subgroup of $G$ is submodular in $G$.

Denote $\mathfrak{s}U$ the class of all strongly supersoluble groups. The following results are obtained.

Theorem 1. Let $G$ be a group. Then the following hold:

1) if $G$ is strongly supersoluble, then every subgroup of $G$ is strongly supersoluble;
2) if $G$ is strongly supersoluble and $N \leq G$, then $G/N$ is strongly supersoluble;
3) if $N_i \leq G$ and $G/N_i$ is strongly supersoluble for $i = 1, 2$, then $G/N_1 \cap N_2$ is strongly supersoluble; 4) if $H_i \leq G, H_i$ is strongly supersoluble, $i = 1, 2$ and $H_1 \cap H_2 = 1$, then $H_1 \times H_2$ is strongly supersoluble;
5) if $G/\Phi(G)$ is strongly supersoluble, then $G$ is strongly supersoluble;
6) the class of groups $\mathfrak{s}U$ is a hereditary saturated formation.

We denote $\mathfrak{A}$ the class of all abelian groups of exponent free from squares of primes.

Theorem 2. The class of all strongly supersolubility groups is a local formation and has a local screen $f$ such that $f(p) = \mathfrak{A}(p-1) \cap \mathfrak{A}$ for any prime $p$.

Theorem 3. Let the group $G = AB$ be the product of nilpotent subgroups $A$ and $B$. If $A$ and $B$ are submodular in $G$, then $G$ is strongly supersoluble.

In Theorem 3 we can’t discard the submodularity of one of subgroups.

Example. In group $G = AB$, where $A \simeq Z_{17}$ and $B \simeq Aut(Z_{17}) \simeq Z_{16}$, the subgroup $A$ is submodular, but the subgroup $B$ is not submodular in $G$. The group $G$ is supersoluble, but not strongly supersoluble. The example also shows that $\mathfrak{s}U \neq \mathfrak{U}$.

Theorem 4. A group $G$ is strongly supersoluble if and only if $G$ is metanilpotent and any Sylow subgroup of $G$ is submodular in $G$.

References