# SINGER CYCLES IN COMPLEX REPRESENTATIONS OF THE GENERAL LINEAR GROUP OVER A FINITE FIELD

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Let $G = PGL(n, q)$ be the projective general linear group of degree $n$ over a finite field of $q$ elements. Let $t \in G$ be a Singer cycle in $G$, that is, an element of order $(q^n - 1)/(q - 1)$ whose preimage in $GL(n, q)$ is irreducible. Let $\phi$ be an irreducible representation of $G$ over the complex numbers. We prove that 1 is an eigenvalue of $\phi(t)$, unless, possibly, the degree of $\phi$ is strictly less than $|t|$, the order of $t$. This answers a question raised by Pablo Spiga (University of Milan in Bicocca). Irreducible representations of $G$ of degree less than $|t|$ are well known, and the inspection yields a more precise answer. Namely, the degree is either $|t| - 1$ or 1, or 3 for the case where $(n, q) = (3, 2)$.

Apart from an intrinsic interest, the result is assumed to be used as a base of induction for studying the occurrence of eigenvalue 1 for other semisimple elements of $G$. The method can probably be used to prove that the minimum polynomial degree of $\phi(t)$ equals $|t|$ with the same exceptions as above. In another direction, one can try to generalize the result to other classical groups. (The case $G = PSL(n, q)$ can be easily deduce to the above result.)

The proof is somehow by induction on the number of divisors of $n$. If $n$ is a prime, the result follows by applying standard results of the Deligne-Lusztig theory of characters of groups of Lie type. The main difficulties arise in performing the induction step. In this situation, that is, when $n$ is not a prime, an essential role in the proof is played by representation theory of groups with cyclic Sylow $p$-subgroup, not only over the complex numbers but also over the ring of $p$-adic integers. The starting point is the fact that the group $T = \langle t \rangle$ contains a cyclic Sylow $p$-subgroup, unless $n = 2$ and $q + 1$ is a 2-power, or $(n, q) = (6, 2)$. However, the reasonuing is not straightforward as the representation theory of groups with cyclic Sylow $p$-subgroup is efficient for analyzing eigenvalues of $p$-elements whereas $t$ is not usually a $p$-element. Exactly this requires realization of the representation in question over the ring of $p$-adic integers, and some use of the theory of projective modules over such rings.