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RESEARCH ARTICLE

SEMILATTICES OF WIDTH 3 WHOSE DECOMPOSABLE ELEMENTS FORM A CHAIN

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Describing semilattices of width 2 modulo chains (totally ordered sets) in [3], the author essentially used the comparability of decomposable elements of such semilattices. It is natural to consider the class of all semilattices of width 3 whose decomposable elements are pairwise comparable. This paper describes semilattices of this class modulo chains. Each one of such semilattices may be constructed as an extension of a chain by a forest whose every tree has width $\leqq 2$. In a sense, the construction is invariant under automorphisms. The main theorems are given in §2.

In §9 examples are furnished; in particular, diagrams of all ordinally indecomposable semilattices of width 2 or 3 having $\leqq 7$ elements and with the considered property are pictured.

Here we use terminology and notation of the theory of ordered sets [1], [2]... . A list of notation and notions useful in this paper follows.

The symbols $\vee$ and $\wedge$ mean disjunction and conjunction, respectively.

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Let E be a set. Then $|E|$ is its cardinality, $\Delta_E = \{(a,a) \in E \times E \mid a \in E\}$.

If $\rho \subset E \times F$, $\sigma \subset F \times G$ are relations, then their composition $\sigma \circ \rho \subset E \times G$ is read from right to left. A disjoint union (sum) of sets E and F is denoted by $E \sqcup F$.

Now let E be an ordered set with the order relation $\leqq$. If $A, B \subset E$, then

$$A \leqq B \overset{\text{def}}{\Longleftrightarrow} \forall a \in A \ \forall b \in B \ (a \leqq b),$$

$$A \gtrless B \overset{\text{def}}{\Longleftrightarrow} \forall a \in A \ \forall b \in B \ (a \leqq b \vee b \leqq a),$$

$$A \parallel B \overset{\text{def}}{\Longleftrightarrow} \forall a \in A \ \forall b \in B \ (a \nleqq b \wedge b \nleqq a),$$

$$[E, A] \overset{\text{def}}{=\!=} \{a \in E \mid \exists b \in A \ (a \leqq b)\},$$

$$[E, A[\overset{\text{def}}{=\!=} [E, A] \setminus A .$$

$[A, E]$ and $]A, E]$ are defined dually. Now $[A, E, B] \overset{\text{def}}{=\!=} [A, E] \cap [E, B]$ and, according to the preceding denotations, $[A, E, B[$, $]A, E, B]$, $]A, E, B[$ are defined.

Further

$$\mathrm{Com}\, A \overset{\text{def}}{=\!=} \{a \in E \mid \{a\} \gtrless A\}$$

$$E\!\downarrow \; = \begin{cases} E \setminus \{a\} & \text{if } a = \inf E \\ E & \text{if } E \text{ has no least element.} \end{cases}$$

Dually, $E\!\uparrow$ is defined. If $A = \{a\}$, $B = \{b\}$, then in the preceding denotations replace A by a, B by b. If E is the chain $\mathbb{Z}$ of integers, $m, n \in \mathbb{Z}$, instead of $[m, \mathbb{Z}, n]$ write $[m, n]$, as usual.

An ordered set E is said to be a <u>forest</u> if for every $a \in E$ the set $[a, E]$ is a chain. A maximal subchain of a forest E will be called a <u>branch</u> of E. A subset A of E is said to be *cofinal (coinitial, convex)* in E if $[E, A] = E$ ($[A, E] = E$, $[A, E, A] = A$, respectively). A subset A of E is called an *antichain* if $\forall a \in A \ (a \parallel A \setminus \{a\})$. The least upper bound of cardinalities of all antichains of E is called *width* of E.

Now let E be a (lower) semilattice. If $a, b \in E$, then the g.l.b. of $\{a, b\}$ is denoted by ab. An element $a \in E$ is called *decomposable* if $a = bc$ for some incomparable elements $b, c \in E$ (i.e. $b \parallel c$). The set of all decomposable elements of E will be denoted by Dec E. If Dec E is a chain, then E is called *single-trunked*. A set Com Dec E will be denoted by Kar E. A set $\mathrm{Piv}\, E \overset{\mathrm{def}}{=\!=} \mathrm{Com}\,\mathrm{Kar}\, E$ will be called a *pivot* of E. If E is single-trunked, then, evidently, $\mathrm{Dec}\, E \subseteq \mathrm{Piv}\, E \subseteq \mathrm{Kar}\, E$ and Piv E is a subchain of E.

A subsemilattice F of a semilattice E is said to be a *filter* of E if $[F, E] = F$. A filter F is called *nodal* if $E \setminus F \leqq F$. An ordinal sum [1] of a family $\{E_\alpha\}_{\alpha \in \Gamma}$ of ordered sets (Γ is a chain) is denoted by $\Sigma_{\alpha \in \Gamma} E_\alpha$. It is known that every semilattice is an ordinal sum of its (ordinally) indecomposable subsemilattices, such a decomposition is unique.

LEMMA 1.1. [3, Lemma 2.1] *A semilattice* E *is indecomposable if and only if* E *is the only nodal filter of* E.

Now let E be a chain. Consider a family $\{A_\alpha\}_{\alpha \in \Gamma}$ of its convex subsets. Let $a \in E$. The set $\{\alpha \in \Gamma \mid a \in A_\alpha\}$ is denoted by Γ_a, the set $\cap \{A_\alpha \mid \alpha \in \Gamma_a\}$ is denoted by $< a >_\Gamma$. Define *density* $d_\Gamma(a)$, *height* $h_\Gamma(a)$, *rank* $r_\Gamma(a)$ of an element a relative to the considered family by the following formulae:

$$d_\Gamma(a) = 1 \iff |< a >_\Gamma| > 1$$

$$d_\Gamma(a) = 0 \iff |\langle a \rangle_\Gamma| = 1$$

$$h_\Gamma(a) = |\Gamma_a|, \quad r_\Gamma(a) = d_\Gamma(a) + h_\Gamma(a)$$

Rank r_Γ of the family $\{A_\alpha\}_{\alpha \in \Gamma}$ is defined as $\sup\{r_\Gamma(a) \mid a \in A\}$. If $A = \bigcup_{\alpha \in \Gamma} A_\alpha$, then the family considered is called a covering of E.

A chain E will be called a δ-chain if it is a dually well-ordered set. If E is not a δ-chain, then define I(E) as the greatest ideal (by inclusion) among ideals of E without greatest elements.

Now let R be a convex subchain of $\mathbb{Z}$. A mapping $x : R \to [0,2] : r \to x_r$ is called an injunction on R if

R1. $\forall r \in R \ (r = \max R \vee r = \min R \Longrightarrow x_r = 0)$,

R2. $\forall r \in R \ (x_r \in 2\mathbb{Z} \iff r \in 2\mathbb{Z})$.

Suppose that $P \subseteq R$ and

R3. $\forall r \in R \ (r \notin P \Longrightarrow r+1,\ r-1 \in P \wedge x_{r-1} = x_{r+1} \neq 0)$.

Then a restriction $y : P \to [0,2]$ of x on P is said to be a mold of x.

2. MAIN THEOREMS.

It is easy to see that a semilattice E is single-trunked and has width 3 iff every component of its ordinal decomposition is either a chain or a single-trunked semilattice of width $\leqq 3$, at least one of its components having width 3. Semilattices of width 2 are single-trunked and have been described in [3]. Therefore, describing single-trunked semilattices of width 3 is reduced to that of indecomposable ones. A construction to obtain any such semilattice is offered below and the process of constructing is realized in five steps: 1) obtaining an initial covering of a chain A, 2) completing the initial covering up to a fundamental covering of A, 3) picking out an intermediate family of subchains of A inscribed in the fundamental covering, 4) obtaining an accompanying monotone mapping φ from a forest into the chain A, 5) forming of a semilattice E as an extension of the chain A along φ.

1) Let $x : R \to [0,2]$ be an injunction on R and $y : P \to [0,2]$ a mold of x. Set $Q = R \cap 2\mathbb{Z}$.

Let a chain A be an ordinal sum of a family $\{C_p\}_{p \in P}$ of chains such that

E1. $\forall p \in P \ (y_p = 2 \implies |C_p| = 1)$

Then a triple

$$(y, \{C_p\}_{p \in P}, A) \tag{2.1}$$

will be called _initial._

For $\alpha \in Q$ set

$$A_\alpha = \bigcup \{C_p \mid p \in P, \ |p-\alpha| \leqq y_p\} \tag{2.2}$$

LEMMA 2.1. _A family_

$$\{A_\alpha\}_{\alpha \in Q} \tag{2.3}$$

is a covering of the chain A.

This covering is said to be an _initial covering corresponding_ to the initial triple (2.1).

2) Let

$$\{A_\alpha\}_{\alpha \in \Gamma}, \tag{2.4}$$

where $Q \subseteq \Gamma$ is a family of convex subchains of A. (Then (2.4) is a covering of A.) Set $\Lambda = \Gamma \setminus Q$. Suppose that the following conditions are satisfied:

E2. If $\Lambda \neq \emptyset$, then a family

$$\{A_\alpha\}_{\alpha \in \Lambda} \tag{2.5}$$

has rank $\leqq 2$.

E3. For $p \in P$ if r_p is rank of the covering

$$\{C_p, C_p \cap A_\alpha\}_{\alpha \in \Lambda} \tag{2.6}$$

of the chain C_p, then

$$\max\{y_p + r_p \mid p \in P\} = 3 \tag{2.7}$$

E4. $\forall \alpha \in \Lambda\ \exists \beta \in Q\ (A_\alpha \subseteq A_\beta)$

E5. $\forall \alpha \in \Lambda\ (|A_\alpha| = 1 \Longrightarrow \exists \beta \in \Gamma \setminus \{\alpha\}\ ([A, A_\alpha] = [A, A_\beta]))$.

E6. $\forall \alpha \in Q\ (\alpha = \max Q \wedge (A_\alpha$ is a δ-chain $\vee \exists \beta \in \Lambda ([A_\beta, A]$ is a δ-chain and is contained in $A_\alpha)) \Longrightarrow \exists \gamma \in \Lambda (A = [A, A_\gamma] \vee \exists a \in A_\alpha \downarrow (\{a\} = \{\max \cup \{A_\lambda \mid \lambda \in \Lambda\}\} = A_\gamma)))$.

Then the covering (2.4) is said to be <u>fundamental</u> and <u>completing</u> the initial covering (2.3).

PROPOSITION 2.2. <u>For every initial covering (2.3) there exists a fundamental covering (2.4) completing the covering (2.3).</u>

3) Consider a family

$$\{\mathcal{D}_\alpha\}_{\alpha \in \Gamma} \tag{2.8}$$

of subchains of the chain A. Suppose that the following conditions are satisfied:

E7. $\forall \alpha \in \Gamma\ (\mathcal{D}_\alpha$ is contained in A_α and is coinitial in $A_\alpha)$.

E8. $\forall\alpha \in \Gamma\ \forall a \in \mathcal{D}_\alpha\ (a = \max A_\alpha \Longrightarrow \exists \beta \in \Gamma \setminus \{\alpha\}\ (a = \max A_\beta \in \mathcal{D}_\beta))$.

E9. $\forall\alpha \in \Gamma\ \forall a \in \mathcal{D}_\alpha\ (a = \max(\bigcup_{\beta \in \Gamma} \mathcal{D}_\beta) \cap A_\alpha \Longrightarrow a = \max A_\alpha)$.

Then the family (2.8) is called an <u>intermediate</u> family <u>inscribed</u> in the fundamental covering (2.4).

PROPOSITION 2.3. <u>In every fundamental covering (2.4) one can inscribe an intermediate family</u>.

4) Let B be a forest with a family $\{B_\alpha\}_{\alpha\in\Gamma}$ of all its branches. Let $\varphi : B \to A$ be a monotone mapping and let the following conditions be satisfied:

E10. $\forall\alpha \in \Gamma\ (\varphi(B_\alpha) = D_\alpha)$.

E11. $\forall\alpha,\beta \in \Gamma\ (B_\alpha \cap B_\beta \neq \emptyset \Longrightarrow [A, A_\alpha] = [A, A_\beta])$.

E12. $\forall\alpha \in \Gamma\ \forall a \in D_\alpha\ (a = \max A_\alpha \Longrightarrow$

$$\exists\beta\in\Gamma\ \ (a = \max A_\beta \in D_\beta \wedge B_\alpha \cap B_\beta = \emptyset))$$

Then the family

$$(\{B_\alpha\}_{\alpha\in\Gamma},\ B,\ \varphi,\ A,\ \{A_\alpha\}_{\alpha\in\Gamma}) \tag{2.9}$$

will be called an <u>admissible family</u> and φ will be called an <u>accompanying</u> mapping.

PROPOSITION 2.4. <u>For every fundamental covering (2.4) there exists an admissible family (2.9)</u>.

Now let (2.9) be an admissible family. Set $E = B \sqcup A$ and define a relation σ_E on E:

$$\sigma_E = \sigma_A \cup \sigma_\beta \cup \bigcup_{\alpha\in\Gamma,\ a\in B} (\sigma_\alpha \cup \Delta_E)\circ(\sigma_a \cup \Delta_E) \tag{2.10}$$

where σ_A and σ_B are the order relations on A and B

respectively and for $\alpha \in \Gamma$, $a \in B$,

$$\sigma_\alpha = B_\alpha \times]\, A_\alpha, A], \quad \sigma_a = [A, \varphi(a)] \times \{a\} \tag{2.11}$$

THEOREM 2.5. The relation σ_E is the order relation on a (lower) indecomposable single-trunked semilattice E of width 3 with a pivot A.

In this case the family (2.9) is said to be a foundation of E and E will be denoted by $E(\{B_\alpha\}_{\alpha\in\Gamma}, B, \varphi, A, \{A_\alpha\}_{\alpha\in\Gamma})$ and will be called an extension of A along φ.

THEOREM 2.6. Every indecomposable single-trunked semilattice of width 3 is an extension of its pivot A along an accompanying mapping φ.

REMARK 2.7. Let (2.9) be an admissible family. From E11, convexity of the chains of the covering (2.4), E12 and E3 it follows that the meet of any three different branches of the forest B is empty, i.e. every tree of B has width $\leqq 2$.

One can find in §9 examples of initial triples and admissible families. Lemma 2.1 and Propositions 2.2-2.4 are proved in §4-6. Proofs of Theorems 2.5, 2.6 are contained in §7 and §8, respectively.

3. CONDITIONS OF ISOMORPHISM.

THEOREM 3.1. Let for $k = 1, 2$ $(Y^k, \{C_p^k\}_{p\in P^k}, A^k)$ be initial triples and $\{A_\alpha\}_{\alpha\in Q^k}$ initial coverings corresponding to them.

Let $\{A^k_\alpha\}_{\alpha \in \Gamma^k}$ *be completions the corresponding initial coverings up to the fundamental ones. Let* $(\{B^k_\alpha\}_{\alpha \in \Gamma^k}, B^k, \varphi^k, A^k, \{A^k_\alpha\}_{\alpha \in \Gamma^k})$ *be admissible families and* $E^k = E(\{B^k_\alpha\}_{\alpha \in \Gamma^k}, B^k, \varphi^k, A^k, \{A^k_\alpha\}_{\alpha \in \Gamma^k})$. *Then* E^1 *is isomorphic to* E^2 *if and only if the following conditions are satisfied:*

I1. *There exists an isomorphism* ζ *of the chain* $\mathbb{Z}$ *onto* $\mathbb{Z}$ *such that* $\zeta(Q^1) = Q^2$, $\zeta(P^1) = P^2$ *and*

$$\forall p \in P^1 (y^2_{\zeta(p)} = y^1_p)$$

I2. *There exists an isomorphism* η *of the chain* A^1 *onto* A^2 *such that*

$$\forall p \in P(\eta(C^1_p) = C^2_{\zeta(p)})$$

I3. *There exists a bijection* μ *of* Γ^1 *onto* Γ^2 *such that restrictions of* μ *and* ζ *on* Q^1 *are equal.*

I4. *There exists an isomorphism* ν *of the forest* B^1 *onto* B^2 *such that* $\forall \alpha \in \Gamma^1 (\nu(B^1_\alpha) = B^2_{\mu(\alpha)}) \wedge \eta \circ \varphi^1 = \varphi^2 \circ \nu$.

This theorem is stated without proof.

4. INITIAL COVERING.

We are going to prove Lemma 2.1. Let (2.1) be an initial triple, $R = [P, \mathbb{Z}, P]$, $x : R \to [0, 2]$ be an injunction on R and y be a mold of x, $Q = R \cap 2\mathbb{Z}$. For $\alpha \in Q$ define A_α by formula (2.2).

LEMMA 4.1. A_α *is a convex subchain of the chain* A.

PROOF. Let $\alpha \in Q$. If $\alpha \in P$, then $C_\alpha \subseteq A_\alpha \neq \emptyset$. If $\alpha \notin P$, then, by R3, $\alpha+1 \in P$, $y_{\alpha+1} \geqq 1 = |\alpha+1-\alpha|$, whence $C_{\alpha+1} \subseteq A_\alpha \neq \emptyset$. Thus $A_\alpha \neq \emptyset$.

Let $p \in P$. Notice that $y_p \geqq |p-\alpha| \Longleftrightarrow \alpha \in [p-y_p, p+y_p]$. Hence

$$\forall p \in P (C_p \subseteq A_\alpha \Longleftrightarrow \alpha \in [p-y_p, p+y_p] \tag{4.1}$$

Let $m, n, p \in P$, $m \leqq n \leqq p$ and $C_m \cup C_p \subseteq A_\alpha$. By virtue of (4.1) $\alpha \in [m-y_m, m+y_m] \cap [p-y_p, p+y_p]$.

Suppose that $C_n \nsubseteq A_\alpha$. Then $m < n < p$ and either $\alpha < n-y_n$ or $n + y_n < \alpha$. Consider the former case. Then $p-y_p \leqq \alpha < n-y_n$. Hence $p-2 \leqq p-y_p < n-y_n \leqq n < p$. Therefore $n = p-1$, $y_n = 0$, $y_p = 2$ which contradict R2. The latter case is considered analogously. Thus $C_n \subseteq A_\alpha$, the latter chain being convex in A.

LEMMA 4.2. <u>The family (2.3) is a covering of the chain</u> A.

PROOF. Given $a \in A$, there exists $p \in P$ such that $a \in C_p$. If $p \in Q$, then $y_p \geqq 0 = |p-p|$ and $C_p \subseteq A_p$. Suppose that $p \notin Q$. According to R1, $R = [Q, \mathbb{Z}, Q]$ and $P \subset R$ implies that $p-1$, $p+1 \in Q$. From R4 it follows that $y_p \geqq 1 = |p-(p \pm 1)|$ and $C_p \subseteq A_{p-1} \cap A_{p+1}$.

COROLLARY 4.3. $\forall \alpha \in Q \downarrow (A_{\alpha-2} \cap A_\alpha \neq \emptyset)$

PROOF. Let $\alpha-2$, $\alpha \in Q$. If $\alpha-1 \in P$, then as in the end of the proof of the previous lemma $C_{\alpha-1} \subseteq A_{\alpha-2} \cap A_\alpha \neq \emptyset$. If $\alpha-1 \notin P$, then by R3, $\alpha, \alpha-2 \in P$ and $y_{\alpha-2} = y_\alpha \neq 0$. From R2 it follows that $y_{\alpha-2} = y_\alpha = 2$. Therefore $y_\alpha \geqq |\alpha-2-\alpha|$ and $C_\alpha \subseteq A_\alpha \cap A_{\alpha-2} \neq \emptyset$, q.e.d.

Now under the hypotheses of this section we state a few more lemmas.

LEMMA 4.4. $\forall p \in P \forall a \in C_p \; (h_Q(a) = y_p + 1)$.

PROOF. Given $p \in P$, we prove that $p \pm y_p \in Q$. If $p \notin 2\mathbb{Z}$, then using R1 and R2 we infer that $p \pm y_p = p \pm 1 \in Q$. If $p \in 2\mathbb{Z}$, then $p \in 2\mathbb{Z} \cap P \subseteq 2\mathbb{Z} \cap R = Q$. If $y_p = 0$, then $p \pm y_p \in Q$. If $y_p = 2$, then, by R1, $p \in Q\uparrow \cap Q\downarrow$ whence $p \pm y_p = p \pm 2 \in Q$. Now if $a \in C_p$, then by reason of (4.1) we obtain

$$h_Q(a) = |\{a \in Q \mid C_p \subseteq A_\alpha\}| = |Q \cap [p-y_p, \; p+y_p]| =$$

$$= |2\mathbb{Z} \cap [p-y_p, \; p+y_p]| = |[p-y_p, \; 2\mathbb{Z}, \; p+y_p]| = y_p + 1 .$$

LEMMA 4.5. $\forall \alpha, \beta \in Q$,

$$\alpha \leqq \beta \Longleftrightarrow [A_\alpha, A] \subseteq [A, A_\beta] \Longleftrightarrow [A_\beta, A] \subseteq [A, A_\alpha] .$$

PROOF. Let $\alpha, \beta \in Q$, $\alpha < \beta$. Then $\alpha + 2 \leqq \beta$. Let $p \in P$ and $C_p \subseteq A_\alpha$. Then, by (2.2), $p \leqq \alpha + y_p \leqq \alpha + 2 \leqq \beta$. If $\beta = p$, then $\beta = \alpha + 2$, $y_\beta = 2$ and either $\beta + 1 \in P$, $y_{\beta+1} = 1$ or $\beta + 2 \in P$, $y_{\beta+2} = 2$. Hence either $C_{\beta+1} \subseteq A_\beta$ or $C_{\beta+2} \subseteq A_\beta$. Therefore

$$\exists m \in P \; (p < m \wedge C_m \subseteq A_\beta)$$

If $p < \beta$, then either $\beta \in P$ and $C_\beta \subseteq A_\beta$ or (by R3) $\beta + 1 \in P$ and $C_{\beta+1} \subseteq A_\beta$. In this case also the latter assertion holds. Thus

$$\alpha < \beta \Longrightarrow [A, A_\alpha] \subset [A, A_\beta] .$$

Dually $\alpha < \beta \Longrightarrow [A_\beta, A] \subset [A_\alpha, A]$ is examined. The converse implications follow from these ones since $2\mathbb{Z}$ and A are totally ordered.

COROLLARY 4.6.

$$\forall \alpha, \beta \in Q\ (A_\alpha \subseteq A_\beta \Longrightarrow \alpha = \beta) \tag{4.2}$$

$$\forall \alpha \in Q\ (|A_\alpha| = 1 \Longrightarrow |Q| = 1) \tag{4.3}$$

PROOF. (4.2) immediately follows from Lemma 4.5. (4.3) follows from Corollary 4.3 and Lemma 4.5.

5. FUNDAMENTAL COVERING.

To prove Proposition 2.2, assume (2.3) to be an initial covering of A.

Consider several cases.

1) Q has no greatest element and

$$\exists p \in P\ (y_p = 2 \vee y_p = 1 \wedge |C_p| > 1) \tag{5.1}$$

In this case set $\Gamma = Q$, $\Lambda = \emptyset$. Check the conditions E2-E6 for the family (2.4). The suppositions evidently imply E2, E4-E6. From (5.1) and E1 it follows that $\max \{y_p + r_p \mid p \in P\} = 3$, consequently, E3 holds.

2) Q has no greatest element and (5.1) does not hold. If $\forall p \in P (y_p = 0)$, then R1-R3 imply that $|Q| = 1$ which contradicts the supposition. Hence

$$\forall p \in P\ (y_p \leqq 1 \wedge y_p = 1 \Longrightarrow |C_p| = 1)$$

and for some $r \in P$ and for some $a \in A$ we have $y_r = 1$ and $C_r = \{a\}$. Set $\Lambda = \{\alpha, \beta\}$ and $A_\alpha = A_\beta = \{a\}$. It is easy to see that E2-E6 are fulfilled.

3) Q has the greatest element q and A_q has the greatest element a. Then we set $\Lambda = \{\alpha, \beta\}$ and $A_\alpha = A_\beta = \{a\}$. E2,

E4-E6 immediately follow. By virtue of R1, R3, Lemma 4.5 we infer that

$$q \in P, \quad r_q = 3, \quad r_q + y_q = 3 .$$

Using E1 we obtain

$$\max\{r_p + y_p \mid p \in P\} = 3 .$$

Thus E3 holds.

4) Q has the greatest element q, A_q has no greatest element. Then $q \in P$ and we set $\Lambda = \{\alpha\}$, $A_\alpha = C_q$. E2, E4-E6 immediately follow. If $a \in A_\alpha$, then $r_q = d_\Lambda(a) + h_\Lambda(a) + 1 = 3 = r_q + y_q$. Hence, by E1, $\max\{y_p + r_p \mid p \in P\} = 3$ and E3 holds. Thus (2.4) is a fundamental covering.

6. INTERMEDIATE FAMILY, ADMISSIBLE FAMILY.

We are going to prove Propositions 2.3 and 2.4. First we prove the following:

LEMMA 6.1. Let (2.1) be an initial triple. Let (2.5) be a family of convex subchains of the chain A with finite rank. Set $\Gamma = Q \sqcup \Lambda$. For $p \in P$ let r_p be rank of the family (2.6). Then rank of the family (2.4) is equal to $\max\{y_p + r_p \mid p \in P\}$.

PROOF. Let $p \in P$, $a \in C_p$. Density, height, rank of a relative to the family (2.6) is denoted by $d_p(a)$, $h_p(a)$, $r_p(a)$, respectively. Replacing the index p by Γ, Q, Λ we obtain density, height and rank of a relative to the corresponding families. By $\langle a \rangle_p$ we denote the set $C_p \cap \langle a \rangle_\Lambda$. Set

$$t = \max\{r_p + y_p \mid p \in P\}$$

Note that for $a \in C_p$

$$h_p(a) = h_\Lambda(a) + 1 \tag{6.2}$$

$$r_p = \max\{d_p(a) + h_p(a) \mid a \in C_p\} \tag{6.3}$$

Using Lemma 4.4 and (6.2) we obtain

$$\begin{aligned} r_\Gamma(a) = h_Q(a) + h_\Lambda(a) + d_\Gamma(a) = y_p + 1 + h_p(a) - 1 + d_\Gamma(a) = \\ = y_p + h_p(a) + d_\Gamma(a) \end{aligned} \tag{6.4}$$

If $d_\Gamma(a) \leqq d_p(a)$, then $r_\Gamma(a) \leqq y_p + r_p$. Now let $d_p(a) < d_\Gamma(a)$. This means that $d_p(a) = 0$, $d_\Gamma(a) = 1$. Hence

$$\forall a \in C_p(r_\Gamma(a) \leqq y_p + r_p + 1 \leqq t + 1) \tag{6.5}$$

Suppose that

$$r_\Gamma(a) = t + 1 \tag{6.6}$$

Then $d_p(a) = 0$, $d_\Gamma(a) = 1$, whence $1 = |< a >_p| = |C_p \cap < a >_\Lambda|$, but $|< a >_\Lambda| \geqq |< a >_\Gamma| > 1$. Therefore, $\{a\} = C_p \cap < a >_\Lambda$ and $\exists m \in P \, \exists \, b \in C_m \cap < a >_\Lambda$ $(m = \min\,]p, P] \vee m = \max[P, p[\,)$.

Consider the case $p < m$. If $y_p < y_m$, then in view of (6.4), (6.6) we obtain $t = r_\Gamma(a) - 1 = y_p + h_p(a) + 1 - 1 = y_p + h_p(a) < y_m + h_m(b) \leqq t$ which leads to a contradiction. Thus $y_m \leqq y_p$. Set $\alpha = p - y_p$. As in the proof of Lemma 4.4, $\alpha \in Q$. From (4.1) it follows that $C_p \subseteq A_\alpha$, but $\alpha = p - y_p < m - y_m$ implies $C_m \nsubseteq A_\alpha$. Hence $A_\alpha \cap C_m = \emptyset$ and $[A, A_\alpha] = [A, C_p]$ due to convexity of A_α in A.

Suppose that $n = \max[P, p[$ and there exists $c \in < a >_\Lambda \cap C_n$. Then as above we infer that $y_n \leqq y_p$, $\beta = p + y_p > n + y_n$ and hence $[A_\beta, A] = [C_p, A]$. Using convexity of C_p in A we obtain $\{a\} = C_p \cap < a >_\Lambda = [A, C_p] \cap [C_p, A] \cap < a >_\Lambda = [A, A_\alpha] \cap [A_\beta, A] \cap \cap < a >_\Lambda = A_\alpha \cap A_\beta \cap < a >_\Lambda \supseteq < a >_\Gamma$ contradicting $d_\Gamma(a) = 1$. If

$[<a>_\Lambda, A] \subseteq [C_p, A]$, then $\{a\} = <a>_\Lambda \cap C_p = <a>_\Lambda \cap [A, C_p] =$
$= <a>_\Lambda \cap [A, A_\alpha] = <a>_\Lambda \cap A_\alpha \supseteq <a>_\Gamma$ which leads to a contradiction. Dually one treats the case $m < p$. Thus (6.6) fails and (6.5) implies that

$$\forall a \in A\,(r_\Gamma(a) \leqq t) \tag{6.7}$$

From (6.1), (6.3), (6.2) and Lemma 4.4 it follows that for some $p \in P$, $a \in P$

$$t = y_p + r_p = h_Q(a) - 1 + h_\Lambda(a) + 1 + d_p(a) = h_\Gamma(a) + d_p(a)\ .$$

If $d_p(a) \leqq d_\Gamma(a)$, then $t \leqq h_\Gamma(a) + d_\Gamma(a) = r_\Gamma(a)$ and with (6.7) it provides $t = r_\Gamma(a)$. Now let $d_\Gamma(a) < d_p(a)$. Then $d_\Gamma(a) = 0$, $d_p(a) = 1$, whence $1 < |C_p \cap <a>_\Lambda| \leqq |<a>_Q \cap <a>_\Lambda| = |<a>_\Gamma|$ which contradicts $d_\Gamma(a) = 0$. Consequently, $r_\Gamma = \max\{r_\Gamma(a) \mid a \in A\} = t$.

COROLLARY 6.2. Rank of a fundamental covering of A is equal to 3.

Now let us prove Proposition 2.3. Suppose that (2.4) is a fundamental covering of A.

Let $\alpha \in \Gamma$. If $|A_\alpha| = 1$ or

$$\exists \beta \in \Gamma \setminus \{\alpha\}\ ([A, A_\alpha] = [A, A_\beta]) \tag{6.8}$$

then set $\mathcal{D}_\alpha = A_\alpha$. Now suppose that $|A_\alpha| > 1$ and (6.8) does not hold. If $\alpha \in Q\!\uparrow$ then set $\mathcal{D}_\alpha = A_\alpha \setminus A_{\alpha+2}$. Owing to Lemma 4.5 $\emptyset \neq \mathcal{D}_\alpha < A_{\alpha+2}$. If $\alpha = \max Q$, then either A_α is a δ-chain, either $[A_\beta, A]$ is a δ-chain and is contained in A_α for some $\beta \in \Lambda$, or these cases fail. In the latter occasion set $\mathcal{D}_\alpha = I(A_\alpha)$. In the former cases there exists (due to E6) $a \in A_\alpha\!\downarrow$ such that $a = \max \cup \{A_\lambda \mid \lambda \in \Lambda\}$ and $\{a\} = A_\beta$ for some $\beta \in \Lambda$. Then set

$\mathfrak{D}_\alpha = [A_\alpha, a[$. Consider the case $\alpha \in \Lambda$. If A_α is a δ-chain, then set $\mathfrak{D}_\alpha = A_\alpha\uparrow$. If A_α is not a δ-chain, then set $\mathfrak{D}_\alpha = I(A_\alpha)$. In all cases, $\mathfrak{D}_\alpha$ is an ideal of the chain A_α, hence E7 holds.

Let $\alpha \in \Gamma$, $a \in \mathfrak{D}_\alpha$ and $a = \max A_\alpha$. From the definition of $\mathfrak{D}_\alpha$ it follows that (6.8) occurs or $|A_\alpha| = 1$. In the former case if $\beta \in \Gamma \setminus \{\alpha\}$ and $[A, A_\alpha] = [A, A_\beta]$, then (6.8) holds for β and therefore $A_\beta = \mathfrak{D}_\beta$ and $a = \max A_\beta \in \mathfrak{D}_\beta$. In the latter case (6.8) holds for α in view of E5. Thus E8 holds.

Prove E9. Denote $\bigcup_{\alpha\in\Gamma} \mathfrak{D}_\alpha$ by $\mathfrak{D}$. Let $\alpha \in \Gamma$, $a \in \mathfrak{D}_\alpha$ and

$$a = \max \mathfrak{D} \cap A_\alpha \tag{6.9}$$

Clearly, $a = \max \mathfrak{D}_\alpha$. If for α (6.8) holds or $|A_\alpha| = 1$, then $a = \max A_\alpha$. Now suppose that $|A_\alpha| > 1$ and (6.8) does not take place. If $\alpha \in Q\uparrow$, then, by the definition of $\mathfrak{D}_\alpha$ and by E7, $a < A_\alpha \cap A_{\alpha+2} \supseteq A_\alpha \cap \mathfrak{D}_{\alpha+2} \subseteq A_\alpha \cap \mathfrak{D}$ contradicting (6.9). Now let $\alpha = \max Q$. If for some $\beta \in \Lambda$, $d \in A_\alpha\downarrow$ we have $\{d\} = A_\beta$, $\mathfrak{D}_\alpha = [A_\alpha, d[$, then $a < d \in \mathfrak{D}_\beta \subseteq \mathfrak{D} \cap A_\alpha$ which contradicts (6.9). If A_α is not a δ-chain and $\mathfrak{D}_\alpha = I(A_\alpha)$, then $\mathfrak{D}_\alpha$ has no greatest element contradicting (6.9). Now consider the case $\alpha \in \Lambda$. If A_α is not a δ-chain, then $\mathfrak{D}_\alpha = I(A_\alpha)$ has no greatest element contradicting (6.9). Let A_α be a δ-chain and $b = \max A_\alpha$, $a < b$. Owing to E4, there exists $\beta \in Q$ such that $A_\alpha \subseteq A_\beta$. Due to Corollary 6.2 there exists the greatest element among $\gamma \in Q$ satisfying the condition $\beta \leqq \gamma \wedge b \in A_\gamma$. Denote this element by γ. First consider the case $\gamma \in Q\uparrow$. Then $b < A_{\gamma+2}$ and $b \in \mathfrak{D}_\gamma = A_\gamma \setminus A_{\gamma+2}$ contradicting (6.9). Now consider the case $\gamma = \max Q$. Suppose that $A_\alpha \subseteq A_\gamma$. Let $[A_\alpha, A]$ be not a δ-chain. Then A_γ is not a δ-chain and $b \in I(A_\gamma)$ since A_α is a δ-chain. If $\mathfrak{D}_\gamma = I(A_\gamma)$, then $b \in \mathfrak{D}_\gamma \subseteq \mathfrak{D}$ contradicting (6.9).

If $\mathcal{D}_\gamma = [A_\gamma, c[$ where $c = \max \bigcup \{A_\lambda \mid \lambda \in \Lambda\}$ and $\{c\} = A_\delta$ for some $\delta \in \Lambda$, then there exists $\mu \in \Lambda$ such that $[A_\mu, A]$ is a δ-chain and is contained in A_γ. Since $[A_\alpha, A]$ is not a δ-chain, we obtain $A_\alpha < A_\mu \leqq A_\delta = \{c\}$, whence $b < c$ and $b \in \mathcal{D}_\gamma$ which contradicts (6.9).

Now assume that $[A_\alpha, A]$ is a δ-chain. It follows from E6 that there exist $\delta \in \Lambda$, $c \in A_\alpha{\downarrow}$ such that $c = \max \bigcup \{A_\lambda \mid \lambda \in \Lambda\}$, $\{c\} = A_\delta$. Then $\mathcal{D}_\gamma = [A_\gamma, c[$. Clearly, $b \leqq c$. If $b = c$, then $b \in \mathcal{D}_\delta$ which contradicts (6.9). If $b < c$, then $b \in \mathcal{D}_\gamma$. The case $A_\alpha \subseteq A_\gamma$ is considered. Now let $A_\alpha \not\subseteq A_\gamma$. Then $A_\alpha \cap A_\gamma$ is an ideal of A_γ. If A_γ is not a δ-chain and $\mathcal{D}_\gamma = I(A_\gamma)$, then $b \in \mathcal{D}_\gamma$ since $A_\alpha \cap A_\gamma$ is a δ-chain. If $\mathcal{D}_\gamma = [A_\gamma, c[$ where $c = \max \bigcup \{A_\lambda \mid \lambda \in \Lambda\}$, $\{c\} = A_\delta \subseteq A_\gamma$ for some $\delta \in \Lambda$, then $b \leqq c$ and we use the previous reasoning. Thus E9 holds. Proposition 2.3 is proved.

To prove Proposition 2.4 suppose (2.8) to be an intermediate family inscribing in a fundamental covering (2.4).

For $\alpha \in \Gamma$ let B_α be a copy of $\mathcal{D}_\alpha$ and φ_α be a corresponding isomorphism. Set $B = \bigcup_{\alpha \in \Gamma} B_\alpha$, $\sigma_B = \bigcup_{\alpha \in \Gamma} \sigma_{B_\alpha}$ where γ_{B_α} is the order relation on B_α. Then σ_B is an order relation on B, with B being a forest with a family $\{B_\alpha\}_{\alpha \in \Gamma}$ of all its branches. Let $\varphi = \bigcup_{\alpha \in \Gamma} \varphi_\alpha$. Then φ is a monotone mapping of B into A. E10, E11 obviously follow. E12 follows from E8. We have got the admissible family (2.9).

7. PROOF OF THEOREM 2.5.

Suppose that (2.9) is an admissible family. The following lemmas provide the proof of Theorem 2.5.

LEMMA 7.1. <u>The relation σ_E defined by the formula (2.10) and (2.11) is an order relation on</u> E .

PROOF. From (2.10) it follows that

$$\sigma_A = A \times A \cap \sigma_E \tag{7.1}$$

and therefore the relations σ_A and σ_E may be denoted equally by $\leqq$. The relation σ_B will be denoted by $\preceq$ and the incomparability relation $B \times B \setminus \sigma_B \cup \sigma_B^{-1}$ will be denoted by $|||$. If $b \in B$, then $\varphi(b)$ will be denoted also by b'. If $a \in \Gamma$, then we shall denote $[A, A_\alpha]$ also by $A_\alpha]$. Let $\alpha, \beta \in \Gamma$; $a \in A$, $b \in B_\alpha$, $c \in B_\beta$. From (2.10) and (2.11) it follows that

$$b \leqq a \Longleftrightarrow A_\alpha < a \tag{7.2}$$

$$a \leqq b \Longleftrightarrow a \leqq b' \tag{7.3}$$

$$b \leqq c \Longleftrightarrow b \preceq c \vee A_\alpha < c' \tag{7.4}$$

$$b \parallel a \Longleftrightarrow b' < a \in A_\alpha \tag{7.5}$$

$$b \parallel c \wedge b' \leqq c' \Longleftrightarrow b \,|||\, c \wedge b' \leqq c' \in A_\alpha \tag{7.6}$$

Since B_α is a branch of the forest B and due to E11, we obtain also

$$b \preceq c \Longrightarrow A_\alpha] = A_\beta] \tag{7.7}$$

Reflexivity of σ_E is evident.

To prove transitivity suppose that $(a, b), (b, c) \in \sigma_E$. We have to show that $(a, c) \in \sigma_E$. Consider the case $a \in B_\alpha$, $b \in B_\beta$, $a, \beta \in \Gamma$. Then either $a \preceq b$ or $A_\alpha < b'$. Since $b' \in A_\beta$ and A_β is convex, we obtain by (7.7) that $A_\alpha] \subseteq A_\beta]$. If $c \in A$, then, by (7.4), either $b \preceq c$ or $A_\beta < c$. In the latter case $A_\alpha < c$ and $a < c$. In the former case if $A_\alpha < b'$, then it follows from monotonicity of φ that $A_\alpha < c'$ and $a < c$. If $a \preceq b$, then $a \preceq c$ and $a \leqq c$ owing to (7.4).

Analogously, one can consider the other cases and the antisymmetry of σ_E.

LEMMA 7.2. E is a lower semilattice relative to the order relation σ_E.

PROOF. Let $a, b \in E$ and $a \| b$. (7.5) and (7.6) imply that, up to the interchange of a and b, only two cases are possible:

1) $a \in B_\alpha$, $\alpha \in \Gamma$, $b \in A$

$$a' < b \in A_\alpha \tag{7.8}$$

2) $a \in B_\alpha$, $b \in B_\beta$, $\alpha, \beta \in \Gamma$, $a \|| b$, $a' \leqq b' \in A_\alpha$.

Prove that in both cases

$$\inf\{a, b\} = a' \tag{7.9}$$

Suppose that 1) holds. (7.3) implies that $a' < a$. Let $c \in E$ and $c \leqq \{a, b\}$. If $c \in A$, then (7.3) yields $c \leqq a'$. If $c \in B_\gamma$, $\gamma \in \Gamma$, then $c \leqq b$ with (7.8) yields $A_\gamma < b \in A_\alpha$. From $c \leqq a$ it follows by (7.4) that either $c \preceq a$ or $A_\gamma < a'$. The former case does not hold by (7.7). The latter case implies $c \leqq a'$ due to (7.2).

Case 2) is considered in a similar way.

LEMMA 7.3. The semilattice E is single-trunked with pivot A.

PROOF. The proof of Lemma 7.2 provides an inclusion $\operatorname{Dec} E \subseteq \varphi(B)$. To prove the converse assume that $\alpha \in \Gamma$, $a \in B_\alpha$. If $a' < b$ for some $b \in A_\alpha$, then using the proof of the preceding lemma we infer that $a' = ab \in \operatorname{Dec} E$. Now let $a' = \max A_\alpha$.

E12 implies that for some $\beta \in \Gamma$ and $c \in B_\beta$ $a' = c' = \max A_\beta$ and $B_\alpha \cap B_\beta = \emptyset$. Hence $a \parallel c$. With a glance to (7.6), $a \parallel c$ and $ac = a' \in \operatorname{Dec} E$ due to the proof of Lemma 7.2. Thus $\operatorname{Dec} E = \varphi(B)$. This implies that $\operatorname{Dec} E \subseteq A$ and $\operatorname{Dec} E$ is a chain. Consequently, E is a single-trunked semilattice. Prove that $A = \operatorname{Piv} E$. Since $\operatorname{Dec} E \subseteq A$, $A \subseteq \operatorname{Kar} E$. Let $a \in B_\alpha \cap \operatorname{Piv} E$, $\alpha \in \Gamma$. If $a' \in A_\alpha \uparrow$, then $\exists b \in A_\alpha (a' < b)$, whence $b \parallel a$ which contradicts $a \in \operatorname{Com} \operatorname{Kar} E \subseteq \operatorname{Com} A$. Hence $a' = \max A_\alpha$. As in the beginning of the proof, there exists $\beta \in \Gamma$ and $b \in B_\beta$ such that $a' = b' = \max A_\beta$, $a \parallel b$, $ab = a' = b'$. Let $c \in \operatorname{Dec} E$. If $c \leqq b'$, then (7.3) implies $c \leqq b$. If $b' < c$, then $A_\beta < c$ and, by (7.2), $b < c$. This contradicts $a \in \operatorname{Piv} E = \operatorname{Com} \operatorname{Kar} E$. Consequently, $\operatorname{Piv} E \subseteq A$. Now let $a \in A$ and $b \in \operatorname{Kar} E$. If $b \in A$, then $a \lesseqgtr b$. Let $\alpha \in \Gamma$, $b \in B_\alpha$. Suppose that $a \parallel b$. Then $b' < a \in A_\alpha$. If $c \in \varphi(B) \cap]b', A_\alpha]$, then $c \parallel b$ and $b \notin \operatorname{Kar} E$. Thus

$$b' = \max A_\alpha \cap \varphi(B).$$

In view of E9 $b' = \max A_\alpha$, contradicting $b' < a \in A_\alpha$. Hence $a \lesseqgtr b$, $a \in \operatorname{Piv} E$. Thus $A \subseteq \operatorname{Piv} E$ and $A = \operatorname{Piv} E$.

LEMMA 7.4. <u>The semilattice</u> E <u>has width</u> 3.

PROOF. Let $\{a, b, c, d\}$ be an antichain in E. Since A is a chain, one of these elements at most belongs to A.

Let $a \in A$, $b \in B_\alpha$, $c \in B_\beta$, $d \in B_\gamma$, $\alpha, \beta, \gamma \in \Gamma$. Using (7.5) and (7.6) we may suppose that

$$b' \leqq c' \leqq d' < a \in A_\alpha \cap A_\beta \cap A_\gamma,$$

$$c' \in A_\alpha \cap A_\beta, \quad d' \in A_\alpha \cap A_\beta \cap A_\gamma,$$

$\alpha \neq \beta \neq \gamma \neq \alpha$. Therefore $r_\Gamma(a) = d_\Gamma(a) + h_\Gamma(a) \geqq 1 + 3 = 4$ which contradicts Corollary 6.2.

Now let $a \in B_\alpha$, $b \in B_\beta$, $c \in B_\gamma$, $d \in B_\delta$, $\alpha, \beta, \gamma, \delta \in \Gamma$. Analogously, we come to $A_\alpha \cap A_\beta \cap A_\gamma \cap A_\delta \neq \emptyset$, $\alpha \neq \beta \neq \gamma \neq \delta \neq \alpha \neq \gamma \neq \beta \neq \delta$ contradicting Corollary 6.2. Hence width of E is ≤ 3. Corollary 6.2 asserts that there exist $\{\alpha, \beta\} \subset \Gamma$ such that either $|A_\alpha \cap A_\beta| > 1$ or there exists $\{\alpha, \beta, \gamma\} \subseteq \Gamma$ such that $A_\alpha \cap A_\beta \cap A_\gamma \neq \emptyset$. In the former case pick $a, b \in A_\alpha \cap A_\beta$ such that $a < b$. According to E7 and $\{B_\alpha\}_{\alpha \in \Gamma}$ being a family of branches of the forest B, one can find $c \in B_\alpha$, $d \in B_\beta$ such that $c' \leqq a$, $d' \leqq a$ and $c \,|||\, d$. From (7.5) and (7.6) it follows that $c \,\|\, b \,\|\, d \,\|\, c$. In the latter case let $a \in A_\alpha \cap A_\beta \cap A_\gamma$. One can find $b \in B_\alpha$, $c \in B_\beta$, $d \in B_\gamma$ such that $\{b', c', d'\} \leqq a$ and $b \,|||\, c \,|||\, d \,|||\, b$. From (7.6) it follows that $b \,\|\, c \,\|\, d \,\|\, b$. Consequently, E is a semilattice of width 3.

LEMMA 7.5. <u>The semilattice</u> E <u>is indecomposable</u>.

PROOF. Let F be a nodal filter of E. First, consider the case $A \cap F = \emptyset$. Since E is nodal, $A < F$. Hence $\mathrm{Dec}\, E < F$ and $F \subseteq \mathrm{Kar}\, E$. Let $a \in B_\alpha \cap F$, $\alpha \in \Gamma$. If $b \in A_\alpha \cap\,]a', E]$, then $a \,\|\, b$ which contradicts $A < F$. Thus $a' = \max A_\alpha$ and as in the proof of Lemma 7.3 there exists $c \in B$ such that $a \,\|\, c$, $ac = a'$. If $c \in F$, then $ac \in F$ which contradicts $A \cap F = \emptyset$. If $c \notin F$, then $a \,\|\, c$ contradicts F to be nodal. Now treat the case $A \cap F \neq \emptyset$. The family (2.4) being a covering of A, the set $\Theta = \{\alpha \in \Gamma \mid A_\alpha \cap F \neq \emptyset\}$ is not empty. Prove that

$$\forall \alpha \in \Theta \; (A_\alpha \subseteq F) \tag{7.10}$$

In fact, if it is not so, then $a \notin F$, $b \in F$ for some $a, b \in A_\alpha$, $\alpha \in \Theta$. Clearly, $a < b$. Since $\mathfrak{D}_\alpha$ is coinitial in A_α, there exists $c \in B_\alpha$ such that $c' \leqq a$. By (7.5), $c \| b$, whence $c \in F \wedge c' = ac \in F$ contradicting $a \notin F$. Thus (7.10) holds.

To prove the equality $\Theta = \Gamma$, suppose that $\alpha \in \Theta$. Using E4,we may assume that $\alpha \in Q$. According to Lemma 4.5, $[\alpha, Q] \subseteq \Theta$. We have to show that $Q \subseteq \Theta$. Assume, ex adverso, that for some $\beta \in Q \cap \Theta$, $\beta - 2 \in Q \setminus \Theta$. This implies, by (7.10)and Corollary 4.3, $A_{\beta-2} \cap F \supseteq A_{\beta-2} \cap A_\beta \neq \emptyset$ which contradicts the definition of Θ. Thus $Q \subseteq \Theta$. Then it follows from (7.10) and Lemma 4.2 that $A \subseteq F$. With A being coinitial in E, we conclude that $E = F$. Thus, by Lemma 1.1, E is indecomposable. Theorem 2.5 has been proved.

8. PROOF OF THEOREM 2.6.

Suppose that E is a single-trunked indecomposable semilattice of width 3 with the pivot A. Set $B = E \setminus A$. $B \neq \emptyset$ whereas E has width 3. Consider a correspondence $\varphi : B \to A : b \mapsto \varphi(b)$ where for $b \in B$

$$\varphi(b) = \max\{a \in A \mid a \leqq b\} \tag{8.1}$$

LEMMA 8.1. φ <u>is a mapping of</u> B <u>into</u> A, <u>with</u>

$$\forall a, b \in B\ (a \leqq b \Longrightarrow \varphi(a) \leqq \varphi(b)) \tag{8.2}$$

$$\forall a \in A\ \forall b \in B\ (a \leqq b \Longleftrightarrow a \leqq \varphi(b)) \tag{8.3}$$

PROOF. Let $a \in B$. If $a \gtrless E$, then $a \in \mathrm{Com\,Kar}\, E = A$ which contradicts $B \cap A = \emptyset$. Hence, for some $b \in E$, $b \| a$. Suppose that $b \in A$. Then

$$ab = \varphi(a) \tag{8.4}$$

In fact, $ab \in \mathrm{Dec}\, E \subseteq A$ and $ab \leqq a$. Let $c \in A$, $c \leqq a$. Since

A is a chain, $c \gtrless b$. $b \leqq c$ implies $b \leqq a$ which is impossible; $c < b$ implies $c \leqq ab$. Consequently,

$$ab = \max\{c \in A \mid c \leqq a\} = \varphi(a).$$

Now let $a \in \operatorname{Com} A$. Then $a \in \operatorname{Com} \operatorname{Dec} E = \operatorname{Kar} E$. *Since* $a \notin \operatorname{Piv} E$, there exists $b \in \operatorname{Kar} E$ such that $b \| a$. Using $b \gtrless \operatorname{Com} \operatorname{Kar} E = A$ we can show, as above, that $ab = \varphi(a)$. Thus φ is a mapping of B into A. (8.2) and (8.3) follows evidently from the definitions.

Further for $a \in B$ $\varphi(a)$ is also denoted by a'.

COROLLARY 8.2. <u>If</u> $a, b \in B$, $a \| b$, <u>then</u> $ab = \min\{a', b'\}$.

PROOF. Since $ab \leqq \{a, b\}$ and (8.3), $ab \leqq \{a', b'\}$. If $a' \leqq b'$, then $a' \leqq \{a, b\}$, whence $a' \leqq ab$ and $a' = ab$.

COROLLARY 8.3. <u>Let</u> $a \in A$, $b \in B$. <u>If</u> $a \| b$, <u>then</u> $ab = b'$.

PROOF. Follows from that of Lemma 8.1.

COROLLARY 8.4. $\varphi(B) = \operatorname{Dec} E$.

PROOF. $\varphi(B) \subseteq \operatorname{Dec} E$ follows from the proof of Lemma 8.1. Conversely, let $a, b \in E$, $a \| b$. We may assume $b \in B$. From Corollaries 8.3 and 8.4 it follows that $ab \in \varphi(B)$. Thus $\operatorname{Dec} E \subseteq \varphi(B)$, q.e.d.

In the set of all subchains of an ordered set $(B, \leqq)$ ($\leqq$ is induced by the order relation $\leqq$ on E) pick out the set M of all subchains α which satisfy the condition

$$\forall a \in \alpha\, (A \cap [a, E] = A \cap [\alpha, E]) \tag{8.5}$$

For $a \in B$ we have $\{a\} \in M$ and the union of an increasing (by inclusion) sequence of elements of M belongs to M. Therefore, by Kuratowski-Zorn's Lemma, $\forall a \in B \, \exists \alpha \in M$ ($\forall a \varepsilon B$, α is maximal in M.).

The set of all maximal elements of M will be denoted by Γ and a chain $\alpha \in \Gamma$ will be denoted by B_α. Thus we have the family

$$\{B_\alpha\}_{\alpha \in \Gamma} \tag{8.6}$$

of subchains of $(B, \leqq)$ such that $B = \cup_{\alpha \in \Gamma} B_\alpha$. For $\alpha \in \Gamma$ set

$$\mathfrak{D}_\alpha = \varphi(B_\alpha) \ ,$$

$$A_\alpha = [D_\alpha, A] \setminus [B_\alpha, E] \ . \tag{8.7}$$

LEMMA 8.5. For every $\alpha \in \Gamma$, A_α is a convex subchain of the the chain A. Also

$$\mathfrak{D}_\alpha \text{ is contained and is coinitial in } A_\alpha \ , \tag{8.8}$$

$$]A_\alpha, A] = A \cap [B_\alpha, E] \ . \tag{8.9}$$

PROOF. Let $\alpha \in \Gamma$, $a \in B_\alpha$. Due to (8.5) we infer that $a < A \cap [a, E] = A \cap [B_\alpha, E]$. This implies $a' < A \cap [B_\alpha, E]$ in accordance with (8.3). Consequently, $\mathfrak{D}_\alpha = \varphi(B_\alpha) < A \cap [B_\alpha, E]$, which implies (8.9) by virtue of (8.7) and also (8.8). Evidently, A_α is convex in A, q.e.d.

Consider the following relation on B:

$$\sigma_B = \cup_{\alpha \in \Gamma} \sigma_E \cap (B_\alpha \times B_\alpha) \tag{8.10}$$

where σ_E is the order relation on E. σ_B will be denoted also by $\leq$ and $B \times B \setminus \sigma_B \cup {}^{-1}\sigma_B$ will be denoted by $|||$.

LEMMA 8.6. The relation $\preceq$ is an order relation on B. For $\alpha \in \Gamma$ the set B_α is a maximal subchain of $(B, \preceq)$.

PROOF. Since $\Delta_B \subseteq \sigma_B \subseteq \sigma_E$, $\preceq$ is reflexive and antisymmetric. Let $a, b, c \in B$, $a \preceq b \preceq c$. Then $a \leqq b \leqq c$ and $a \leqq c$. For some $\alpha, \beta \in \Gamma$ we have $a, b \in B_\alpha$; $b, c \in B_\beta$. Hence, by (8.5), $A \cap [a, E] = A \cap [B_\alpha, E] = A \cap [b, E] = A \cap [B_\beta, E] = A \cap [c, E]$. Therefore, $\{a, c\} \in M$ and for some $\gamma \in \Gamma$ $\{a, c\} \subseteq B_\gamma$. Hence $a \preceq c$. Thus $\preceq$ is an order relation on B.

It is clear that B_α is a subchain of $(B, \preceq)$. To prove its maximality, let $a \in B \setminus B_\alpha$, $B_\alpha^\circ = B_\alpha \cup \{a\}$. Suppose that $(B_\alpha^\circ, \preceq)$ is a chain. Then $(B_\alpha^\circ, \preceq)$ is a chain and for some $\beta \in \Gamma$ $a \in B_\beta$ and $B_\alpha \cap B_\beta \neq \emptyset$. Using (8.5), one can examine the equality $A \cap [a, E] = A \cap [B_\alpha^\circ, E]$. Hence $B_\alpha^\circ \in M$ contradicting maximality of B_α in M.

LEMMA 8.7. Let $\alpha, \beta \in \Gamma$, $a \in B_\alpha$, $b \in B_\beta$, $a \leqq b$, $b' \in A_\alpha$. Then $[A, A_\alpha] = [A, A_\beta]$.

PROOF. From (8.7) it follows that $[A, A_\alpha] = A \setminus [B_\alpha, E]$. Therefore, it is sufficient to prove that $A \cap [a, E] = A \cap [b, E]$. $a \leqq b$ implies that $A \cap [b,E] \subseteq A \cap [a,E]$. To prove the converse, suppose that $c \in A$, $a \leqq c$. Then, by (8.9), $A_\alpha < c$. Let $b \nleqq c$. If $c < b$, then due to (8.3) $c \leqq b'$ and $A_\alpha < b'$ violating the supposition $b' \in A_\alpha$. If $c \| b$, then, by virtue of Corollary 8.3, $a \leqq bc = b'$, whence $b' \in A \cap [a,E] = A \cap [B_\alpha, E]$ which contradicts $b' \in A_\alpha$ and (8.9). Thus $b \leqq c$ and the lemma is proved.

The order relation on A will be denoted by σ_A as in §7. Also, we use the earlier denotation $A_\alpha]$ for $[A, A_\alpha]$.

LEMMA 8.8. Under the above denotations the assertions (7.1)-(7.7) hold.

PROOF. (7.1) is evident. Let $a \in A$, $b \in B_\alpha$, $c \in B_\beta$; $\alpha, \beta \in \Gamma$. (7.3) follows from (8.3), (7.7) follows from Lemma 8.7, (7.2) follows from (8.5), (8.7) and Lemma 8.5; (7.5) follows from (7.2) and (7.3). Now let $b \leqq c$. By (8.2), $b' \leqq c'$. In view of (8.8) $b' \in A_\alpha$. Therefore either $A_\alpha < c'$ or $c' \in A_\alpha$. In the latter case, according to Lemma 8.7, $A_\alpha] = A_\beta]$ and $A \cap [b, E] = A \cap [c, E]$, wherefrom for some $\gamma \in \Gamma$ $\{b, c\} \subseteq B_\gamma$ and, by (8.10), $b \preceq c$. Hence $b \leqq c \Longrightarrow b \preceq c \vee A_\alpha < c'$. The converse follows from (8.10) and (7.2). Thus (7.4) holds. (7.6) follows from (7.4). Lemma 8.8 is proved.

This lemma implies that the relations σ_E, σ_A, σ_B satisfy the condition (2.10).

LEMMA 8.9. $(B, \preceq)$ is a forest with the family (8.6) of all its branches.

PROOF. Let $a, b, c \in B$, $a \parallel\!| c$, $c \preceq \{a, b\}$. Then for some $\alpha, \beta \in \Gamma$ $c, a \in B_\alpha$, $c, b \in B_\beta$ and $c \leqq \{a, b\}$. We may assume that $a' \leqq b'$. (7.7) implies $A_\alpha] = A_\beta]$. From $a' \in A_\alpha$ and the convexity of A_α in A it follows that $b' \in A_\alpha$. Then, by (7.6), $a \parallel b$. Consequently, by Corollary 8.2, $c \leqq ab = a'$, whence $a' \in A \cap [c, E] = A \cap [B_\alpha, E]$ which contradicts Lemma 8.5. Thus $(B, \preceq)$ is a forest.

From Lemma 8.6 it follows that (8.6) is a family of its branches. Prove absence of other branches. Let $B^{\mathcal{O}}$ be a branch of the forest B. Let $a, b \in B^{\mathcal{O}}$, $a \preceq b$. We may assume that for some $\alpha \in \Gamma$ $a, b \in B_\alpha$ and $a \leqq b$. Owing to (8.5) we deduce $A \cap [a,E] = A \cap [b,E]$. Hence

$$\forall a \in B^{\circ}(A \cap [a,E] = A \cap [B^{\circ},E])$$

Consequently, $B^{\circ} \in M$ and for some $\beta \in \Gamma$, $B^{\circ} \subseteq B_\beta$. The maximality of B° yields $B^{\circ} = B_\beta$.

LEMMA 8.10. Let $\alpha \in \Gamma$, $a \in B_\alpha$. Then

$$a \in \mathrm{Kar}\,E \Longleftrightarrow a' = \max A_\alpha \tag{8.11}$$

$$a \in \mathrm{Kar}\,E \Longrightarrow \exists \beta \in \Gamma \exists b \in B_\beta\,(a \| b \wedge a' = b' = \max A_\beta) \tag{8.12}$$

PROOF. Let $\alpha \in \Gamma$, $a \in B_\alpha \cap \mathrm{Kar}\,E$. Then $a \in \mathrm{Com\,Dec}\,E \subseteq \mathrm{Com\,Com\,Com\,Dec}\,E = \mathrm{Com\,Piv}\,E = \mathrm{Com}\,A$. Hence $a \gtrless A$. If $a' \neq \max A_\alpha$, then $a \gtrless A$ contradicts (7.5). Thus $a' = \max A_\alpha$. Conversely, if $a' = \max A_\alpha$, then, by (8.5) and (8.9), $a \gtrless [A, A_\alpha] \cup A \cap [a,E] = [A, A_\alpha] \cup A \cap [B_\alpha, E] = [A, A_\alpha] \cup]A_\alpha, A] = A$. Hence $a \in \mathrm{Com}\,A \subseteq \mathrm{Com\,Dec}\,E = \mathrm{Kar}\,E$. (8.11) holds.

Again let $a \in B_\alpha \cap \mathrm{Kar}\,E$. As in the proof of Lemma 8.1, there exists $b \in B \cap \mathrm{Kar}\,E$ such that $a \| b$, $a' = ab = b'$. (8.11) implies that $b' = \max A_\beta$ where $b \in B_\beta$.

LEMMA 8.11. $\forall \alpha \in \Gamma \forall a \in B_\alpha\,(a' = \max A_\alpha \Longrightarrow \exists \beta \in \Gamma \exists b \in B_\beta\,(a' = b' = \max A_\beta \wedge B_\alpha \cap B_\beta = \emptyset)$.

PROOF. Suppose that $\alpha \in \Gamma$, $a \in B_\beta$, $a' = \max A_\alpha$. According to Lemma 8.10, $\exists \beta \in \Gamma\ \exists b \in B_\beta\,(ab = a' = b' = \max A_\beta$, $a \| b)$. From (7.6) it follows that $a \||| b$, whence $\alpha \neq \beta$. Assume that $c \in B_\alpha \cap B_\beta$. Since B is a forest, $\{a,b\} \preceq c$, therefore $\{a,b\} \leqq c$ and, by (8.2), $c' = a' = b'$. Using Lemma 8.10, we obtain $\exists \gamma \in \Gamma \exists d \in B_\gamma (c \| d \wedge d' = c' = \max A_\gamma)$

Clearly, $\gamma \notin \{\alpha, \beta\}$. Assume that $e \in B_\alpha \cap B_\gamma$. With B being a forest, $e \in B_\alpha \cap B_\beta \cap B_\gamma$. Hence $\{c, d\} \leqq e$, $e' = d' = \max A_\gamma$. Again, using Lemma 8.10,we find $\delta \in \Gamma$, $f \in B_\delta$ such that $e \| f$, $e' = f' = \max A_\delta$, $\delta \notin \{\alpha, \beta, \gamma\}$. We infer that $a' = b' = c' = d' = e' = f'$. It is easy to see that $\{a, b, d, f\}$ is an antichain in $(B, \preceq)$ and, by (7.6), an antichain in E which contradicts E having width 3.

LEMMA 8.12. Let $\alpha \in \Gamma$, $a \in B_\alpha$, $a' = \max \varphi(B) \cap A_\alpha$. Then $a' = \max A_\alpha$.

PROOF. Under hypotheses of the lemma suppose that $a' \neq \max A_\alpha$. Then, by (8.11), $a \notin \operatorname{Kar} E$, wherefrom it follows that for some $b \in \operatorname{Dec} E$ $a \| b$. On account of (7.5), we infer that $a' < b \in A_\alpha$ but due to Corollary 8.4 $b \in \varphi(B)$. This leads to a contradiction. Hence $a' = \max A_\alpha$.

LEMMA 8.13. Rank of the family

$$\{A_\alpha\}_{\alpha \in \Gamma} \tag{8.13}$$

of convex subchains of A equals 3.

PROOF. Let $\{\alpha_k\}_{k=1}^4 \subseteq \Gamma$ and $a \in \bigcap_{k=1}^4 A_{\alpha_k}$. Since $\mathcal{B}_{\alpha_k}$ is coinitial in A_{α_k} and B is a forest, we can find $a_k \in B_{\alpha_k}$, $k \in [1, 4]$, such that $\{a'_k\}_{k=1}^4 \leqq a$ and $\{a_k\}_{k=1}^4$ is an antichain in $(B, \preceq)$. Owing to (7.6) this is an antichain of E contradicting the hypotheses. Hence $\forall a \in A \, (h_\Gamma(a) \leqq 3)$. By similar arguments one can prove that $\max\{r_\Gamma(a) \mid a \in A\} = 3$. Lemma is proved.

Now consider a relation $\ll$ on Γ : if $\alpha, \beta \in \Gamma$, then

$$\alpha \ll \beta \overset{\text{def}}{\Longleftrightarrow} A_\alpha \subseteq A_\beta .$$

The relation $\ll$ is, evidently, a quasiorder relation with a kernel $\varepsilon = \{(\alpha, \beta) \in \Gamma \times \Gamma \mid A_\alpha = A_\beta\}$. The induced order relation on Γ/ε will be denoted also by $\ll$. Every chain in $(\Gamma/\varepsilon, \ll)$ has three elements at most by reason of Lemma 8.13. In every maximal ε-class pick an element, the set of those denoted by Q.

LEMMA 8.14. The family

$$\{A_\alpha\}_{\alpha \in Q} \tag{8.14}$$

is a covering of the chain A.

PROOF. Set $\Lambda = \Gamma \setminus Q$. We see from above that E4 holds and

$$\forall \alpha \in Q \, \forall \beta \in \Gamma \, (A_\alpha \subseteq A_\beta \Longrightarrow A_\alpha = A_\beta) \tag{8.15}$$

Due to E4 it suffices to prove that the family (8.13) is a covering of A. Suppose that $a \in A \setminus \bigcup_{\alpha \in \Gamma} A_\alpha$. From (7.5) it follows that $a \in \operatorname{Com} E$, whence a filter $[a, E]$ is nodal and equals E in view of the indecomposability of E. Hence $a = \min E$. Since width of E equals 3, $\{a\} \neq E$. Consider a set $]a, E] \neq \emptyset$. If $a \in \operatorname{Dec} E$, then, by Corollary 8.4 and (8.8), $a \in \varphi(B) \subseteq \bigcup_{\alpha \in \Gamma} A_\alpha$ contradicting the hypothesis. If $a \notin \operatorname{Dec} E$, then $]a, E]$, is a proper nodal filter of E contradicting the indecomposability of E. Thus $A = \bigcup_{\alpha \in \Gamma} A_\alpha$, q.e.d.

Consider a relation $\leqq$ on Q: for $\alpha, \beta \in Q$

$$\alpha \leqq \beta \Longleftrightarrow [A_\beta, A] \subseteq [A_\alpha, A] \Longleftrightarrow$$

$$]A_\beta, A] \subseteq]A_\alpha, A] \Longleftrightarrow [A, A_\alpha] \subseteq [A, A_\beta]$$

$$\Longleftrightarrow [A, A_\alpha[\subseteq [A, A_\beta[\; . \tag{8.16}$$

It is easy to check $\leqq$ to be a total order relation on Q.

LEMMA 8.15. The chain $(Q, \leqq)$ is isomorphic to a subchain of the chain $\mathbb{Z}$ of integers.

PROOF. It is sufficient to verify the finiteness of a bounded sequence in $(Q, \leqq)$. Let

$$\ldots < \alpha_n < \ldots < \alpha_1 \qquad (8.17)$$

be an infinitely decreasing sequence bounded from below by $\alpha \in Q$. Consider a filter $F = \cup_{n=1}^{\infty}[A_{\alpha_n}, E]$ of E. Suppose that $A_\alpha \subseteq F$. Then for every $a \in A_\alpha$ one can fix $n \in \mathbb{N}$ such that $a \in [A_{\alpha_n}, E]$. On account of the convexity of A_{α_n} we infer that either $a \in A_{\alpha_n}$ or $A_{\alpha_n} < a$. Since, by (8.16), $[A_{\alpha_n}, A] \subset [A_\alpha, A]$, in the latter case $A_{\alpha_n} \subset A_\alpha$ contradicting (8.15). Hence $a \in A_{\alpha_n}$. Nay, a similar way leads to $\forall m > n\ (a \in A_{\alpha_m})$ which is contrary to $h_\Gamma(a) \leqq 3$ due to Lemma 8.13. Thus $A_\alpha \nsubseteq F$ and $F \neq E$. The indecomposability of E implies that F is not a nodal filter, thereby

$$\exists n \in \mathbb{N}\ \exists a \in A_{\alpha_n}\ \exists b \in E \setminus F\ (a \parallel b).$$

Therefore, $\exists \beta \in \Gamma\ (b \in B_\beta \wedge \forall m \in \mathbb{N}\ (b' < A_{\alpha_m}))$. From (7.5) it follows that $b' < a \in A_\beta$. (8.16) implies that $\forall m > n\ ([A_{\alpha_n}, A] \subset [A_{\alpha_m}, A])$. Therefore, for $m > n$ either $a \in A_{\alpha_m}$ or $A_{\alpha_m} < a$. The latter leads to $A_{\alpha_m} \subset A_\beta$ contradicting (8.15). Thus $a \in \bigcap_{m>n} A_{\alpha_m}$ contradicting Lemma 8.13. Hence, an infinitely decreasing sequence in Q has no lower bound in Q. Analogously, one can prove the dual statement. The lemma is proved.

Now we may assume that Q is a convex subchain of the chain $2\mathbb{Z}$. Set $R = [Q, \mathbb{Z}, Q]$. Recall that for $a \in A$ $Q_a = \{a \in Q \mid a \in A_\alpha\}$. On account of Lemma 8.14, $Q_a \neq \emptyset$.

LEMMA 8.16. <u>For every</u> $a \in A$, Q_a <u>is a convex subchain of</u> Q.

PROOF. Let $\alpha \in \Gamma$. The convexity of A_α implies that

$$A_\alpha = [A, A_\alpha] \cap [A_\alpha, A] \tag{8.18}$$

Suppose that $a \in A$, $\alpha, \beta \in Q_a$, $\gamma \in Q$ and $a \leqq \gamma \leqq \beta$. Using (8.16) we obtain

$$a \in [A, A_\alpha] \cap [A_\beta, A] \subseteq [A, A_\gamma] \cap [A_\gamma, A] = A_\gamma .$$

Thus $\gamma \in Q_a$. Lemma is proved.

From Lemmas 8.13 and 8.14 it follows that for $a \in A$ $1 \leqq h_Q(a) \leqq 3$. Hence $Q_a = [\alpha, 2\mathbb{Z}, \alpha + 2h_Q(a)-2]$. Set $\psi(a) = \alpha + h_Q(a)-1$ (i.e. the middle of the segment $[Q_a, \mathbb{Z}, Q_a]$). We have a mapping $\psi: A \to R$. Thus for every $a \in A$

$$Q_a = [\psi(a)-h_Q(a)+1,\ 2\mathbb{Z},\ \psi(a) + h_Q(a)-1] \tag{8.19}$$

LEMMA 8.17. ψ <u>is a monotone mapping of the chain</u> A <u>into</u> R.

PROOF. Let $a, b \in A$ and $a < b$. If $\alpha \in Q_a \setminus Q_b$ then $A_\alpha < b$, hence $\forall \beta \in Q_b$ $([A, A_\alpha] \subset [A, A_\beta])$. By (8.16), it means that $Q_a \setminus Q_b < Q_b$. If $\alpha \in Q_b \setminus Q_a$, then $a < A_\alpha$ and $\forall \beta \in Q_a$ $([A_\alpha, A] \subset [A_\beta, A]$. Therefore, by (8.16), $Q_a < Q_b \setminus Q_a$. Consequently,

$$[Q_b \mathbb{Z}] \subseteq [Q_a, \mathbb{Z}] \wedge [\mathbb{Z}, Q_a] \subseteq [\mathbb{Z}, Q_b] . \tag{8.20}$$

The equalities simultaneously hold only when $Q_a = Q_b$. From (8.20) it follows that $\psi(a) \leqq \psi(b)$ since these elements are the middles of the corresponding segments $[Q_a, \mathbf{Z}, Q_a]$ and $[Q_b, \mathbf{Z}, Q_b]$. The lemma is proved.

LEMMA 8.18. $\forall a, b \in A\ (\psi(a) = \psi(b) \Longleftrightarrow Q_a = Q_b)$.

PROOF. follows from the proof of the preceding lemma.

LEMMA 8.19. $\forall \alpha \in Q{\downarrow}(A_{\alpha-2} \cap A_\alpha \neq \emptyset)$.

PROOF. Let F be a filter of E generated by A_α. By (8.16), $[A_\alpha, A] \subset [A_{\alpha-2}, A]$, whence $F \neq E$. Since E is indecomposable, F is not a nodal filter. Therefore, there exist $a \in E \setminus F$, $b \in A_\alpha$ such that $a \| b$. For some $\beta \in \Gamma$ we have $a \in B_\beta$ and $a' < b \in A_\beta$. From E4 (which holds by the definition of Q) it follows that for some $\gamma \in Q$, $A_\beta \subseteq A_\gamma$. $a' \notin A_\alpha$ implies $[A_\alpha, A] \subset [A_\gamma, A]$ and $\gamma < \alpha$, $b \in A_\alpha \cap A_\gamma$. Hence $\gamma, \alpha \in Q_b$, $\gamma \leqq \alpha - 2 < \alpha$ and, by Lemma 8.16, $\alpha - 2 \in Q_b$, $b \in A_{\alpha-2} \cap A_\alpha$. The lemma is proved.

Set $P = \psi(A)$. From the definition of ψ it follows that $P \subseteq R$. For $p \in P$ set $C_p = \psi^{-1}(p)$. The monotonicity of ψ implies that $A = \sum_{p \in P} C_p$. For $p \in P$ set

$$y_p = h_Q(a) - 1 \qquad (8.21)$$

where $\psi(a) = p$. In view of Lemma 8.18 we have a mapping $y : p \rightarrow [0, 2] : p \mapsto y_p$.

LEMMA 8.20. $\forall r \in R\ (r \notin P \Longrightarrow r+1,\ r-1 \in P \wedge y_{r+1} = y_{r-1} \neq 0)$.

PROOF. Prove that P is coinitial and cofinal in R. In fact, let $p = \min P$. Then, by Lemma 8.17, for some $a \in A$ $\psi(a) = p$ and

$$\forall b \in [A, a]\ (\psi(b) = p \wedge Q_a = Q_b).$$

Suppose that $h_Q(a) > 1$ and denote $p - h_Q(a) + 1$ by α, $p + h_Q(a) - 1$ by β. Then $\alpha < \beta$ and $Q_a = [\alpha, 2\mathbb{Z}, \beta]$ due to (8.19). Let $b \in A_\alpha \setminus A_\beta$. Owing to (8.16), $a \in A_\beta$ implies $b < a$. Then $\beta \notin Q_b$ contradicts $Q_a = Q_b$. Thus $h_Q(a) = 1$ and $Q_a = \{p\} \subseteq Q$. From the proof of Lemma 8.17 it follows that $[Q_a, Q] = Q$, whence $p = \min Q = \min R$. Dually, if $p = \max P$, then $p = \max R$. So $[P, R, P] = R$.

Now suppose that $p \in P\!\uparrow$, $a \in C_p$, $b \in C_{p^*}$ where $p^* = \min\,]p, P]$. By Lemma 8.18, $Q_a \neq Q_b$. Suppose that $\alpha, \beta, \gamma, \delta \in Q$, $Q_a = [\alpha, 2\mathbb{Z}, \beta]$, $Q_b = [\gamma, 2\mathbb{Z}, \delta]$. By the proof of Lemma 8.17, $\alpha \leqq \gamma$, $\beta \leqq \delta$ and $\alpha < \gamma \vee \beta < \delta$. Consider the case $\alpha < \gamma$. Then $b \notin A_\alpha$ and $[A, C_p] \subseteq [A, A_\alpha]$, whence $[A, A_\alpha] = [A, C_p] < b$. From (8.16) it follows that there exists $c \in [A, A_{\alpha+2}] \setminus [A, A_\alpha]$. If $A_{\alpha+2} < b$, then, by Lemma 8.17, $p = \psi(a) < \psi(c) < \psi(b) = p^*$ which contradicts the definition of p^*. Thus $\alpha + 2 \in Q_b$ due to Lemma 8.19. Hence $\gamma = \alpha + 2$. Dually, if $\beta < \delta$, then $\delta = \beta + 2$. Consequently, we have three alternatives:

1) $Q_a = [\alpha, 2\mathbb{Z}, \beta]$, $Q_b = [\alpha, 2\mathbb{Z}, \beta + 2]$.

2) $Q_a = [\alpha, 2\mathbb{Z}, \beta]$, $Q_b = [\alpha + 2, 2\mathbb{Z}, \beta]$.

3) $Q_a = [\alpha, 2\mathbb{Z}, \beta]$, $Q_b = [\alpha + 2, 2\mathbb{Z}, \beta + 2]$.

In cases 1) and 2)

$$p^* = \frac{\alpha + \beta + 2}{2} = \frac{\alpha + \beta}{2} + 1 = p + 1, \quad |h_Q(a) - h_Q(b)| = 1.$$

In case 3)

$$p^* = \frac{\alpha+2+\beta+2}{2} = \frac{\alpha+\beta}{2} + 2 = p + 2,\ h_Q(a) = h_Q(b)\ .$$

Here $p + 1 \notin P$ and, by (8.21), $y_p = y_{p+2}$. Suppose that $y_p = 0$. Then $h_Q(a)=h_Q(b)=1$, $Q_a=\{\alpha\}$, $Q_b=\{\alpha+2\}$. By lemma 8.19, there exists $c \in A_\alpha \cap A_{\alpha+2}$. Since $a < A_{\alpha+2}, A_\alpha < b \in A_{\alpha+2}$ we obtain $a < c < b$, whence $p = \psi(a) \leqq \psi(c) \leqq \psi(b) = p^*$. If $p=\psi(c)$, then $Q_a = Q_c$, which contradicts $\alpha+2 \in Q_c$ if $\psi(c) = p^*$, then $Q_c = Q_b$, which contradicts $\alpha \in Q_c$. $p<\psi(c)<p^*$ contradicts the definition of p^*. Thus $y_p \neq 0$ and Lemma 8.20 is proved. Now define a mapping $x:R \to [0,2]: r \to x_r$ in such a way: for $r \in R$

$$x_r = \begin{cases} y_r & \text{if } r \in p \\ & \\ Y_{r-1} & \text{if } r \in R/P\ . \end{cases} \qquad (8.22)$$

Due to Lemma 8.20 the definition is correct.

LEMMA 8.21. <u>The mapping</u> x <u>is an injunction on</u> R, <u>the mapping</u> y <u>is a mold of</u> x, <u>the triple</u> $\{y,\{C_p\}_{p \in P},A\}$ <u>is initial</u>.

PROOF. R3 follows from Lemma 8.20. Suppose that $r = \min R$. Then P is bounded from below; from the beginning of the proof of Lemma 8.20 it follows that $r = \min P$ and for $a \in C_r, 1 = h_Q(a) = y_r+1$, whence $x_r = y_r = 0$. Dually, if $r = \max R$. Thus R1 holds.

Now prove R2. Suppose that $r \in P, a \in C_r, \alpha=\min Q_a$. From the definition of ψ it follows that $r \in 2\mathbb{Z} \Leftrightarrow r = \psi(a) = \alpha+h_Q(a)-1 \in 2\mathbb{Z} \Leftrightarrow \alpha+y_r \in 2\mathbb{Z} \Leftrightarrow y_r \in 2\mathbb{Z} \Leftrightarrow x_r \in 2\mathbb{Z}$. Now suppose that $r \in R \setminus P$. Then $r-1, r+1 \in P$ and $x_r = y_{r-1}-1$. According to the preceding statement, we obtain

$$r \in 2\mathbb{Z} \Leftrightarrow r-1 \notin 2\mathbb{Z} \Leftrightarrow y_{r-1} \notin 2\mathbb{Z} \Leftrightarrow x_r \in 2\mathbb{Z}\ .$$

Thus R2 holds. To complete the proof check E1. Let $p \in P, y_p = 2$. For $a \in C_p$ we have $h_Q(a) = 3$ which implies $d_\Gamma(a) = 0$ due to Lemma 8.13. Hence $|C_p| \leqq |\ \langle a\rangle_Q\ | = 1$. E1 holds.

Lemma 8.21 is proved.

LEMMA 8.22. <u>The family (8.12) is a fundamental covering</u>

of A completing the initial covering (8.14).

PROOF. E4 has been proved above. E5 follows from Lemma 8.11. It follows from Lemma 8.13 that the family (2.6) has a finite rank. Using Lemmas 6.1 and 8.13, we infer that E3 holds. From E3 it is easy to deduce E2.

Now prove E6. Let $\alpha = \max Q$, $|A_\alpha| > 1$. Suppose that A_α is a δ-chain. Then there exists $a \in B$ such that

$$a' = \max \varphi(B) \cap A_\alpha \tag{8.23}$$

Also for some $b \in B_\alpha$ $\varphi(b) = \max \mathcal{B}_\alpha$. If $a' = b'$, then by Lemma 8.12, $a' = \max A_\alpha$. In view of Lemma 8.11, there exists $\beta \in \Gamma \setminus \{\alpha\}$ such that $A = [A, A_\alpha] = [A, A_\beta]$. Clearly, $\beta \in \Lambda$. Now let $b' < a'$. Then for some $\beta \in \Gamma \setminus \{\alpha\}$ $a \in B_\beta$. $[a', A_\beta] \subseteq A_\alpha$ implies that $a' = \max \varphi(B) \cap A_\beta$. Hence, by Lemma 8.12, $a' = \max A_\beta$. According to Lemma 8.11, there exists $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ such that for some $c \in B_\gamma$ $c' = a' = \max A_\gamma$. From Lemma 8.13 it follows that $\{a'\} = A_\alpha \cap A_\beta \cap A_\gamma$. Since $[A, A_\beta] = [A, A_\gamma] \subseteq$ $\subseteq [A, A_\alpha]$ and $a' \in A_\alpha\!\downarrow$ we obtain $A_\beta = \{a'\} \vee A_\gamma = \{a'\}$. Assume for the definiteness sake the former case. Then $\beta \in \Lambda$. Since A_α is a δ-chain, there exists $d = \max \bigcup \{A_\lambda \mid \lambda \in \Lambda\} = \max A_\zeta$ for some $\zeta \in \Lambda$. It is clear that $a' \leqq d$. By $[a', A, d] \subseteq A_\alpha$ and (8.23), $a' = \max \varphi(B) \cap [A, A_\zeta]$. Hence $\mathcal{B}_\zeta \leqq a'$ and, by the convexity of A_ζ, $a' \in A_\zeta$. Suppose that $a' < d$. Then $\zeta \notin \{\beta, \gamma\}$ and $a' \in A_\gamma \cap A_\beta \cap A_\gamma \cap A_\zeta$ which contradicts Lemma 8.13. Thus $a' = d$, $\zeta = \beta$. Consequently,

$$\exists \beta \in \Lambda([A, A_\beta] = A \vee \exists a \in A_\alpha\!\downarrow (\{a\} = A_\beta = \{\max \bigcup \{A_\lambda \mid \lambda \in \Lambda\}\})).$$

Now let $[A_\beta, A]$ for some $\beta \in \Lambda$ be a δ-chain contained in A_α. Then $\varphi(B_\beta) \subseteq \varphi(B) \cap A_\alpha$, therefore the latter set has the greatest element a' for some $a \in B$. Repeating the preceding argument, we complete the proof of E 6.

Thus (8.13) is a fundamental covering of A, completing the initial covering (8.14), q.e.d.

Now consider the family (2.8), where for $\alpha \in \Gamma$ $\mathfrak{D}_\alpha = \varphi(B_\alpha)$. Prove it to be an intermediate family inscribed in the covering (8.13). E7 follows from (8.8), E8 follows from Lemma 8.11, E9 follows from 8.12.

Under the previous denotations, prove that the family (2.9) is admissible. In fact, by Lemma 8.9, the ordered set $(B, \preceq)$ is a forest with the family (8.6) of all its branches. It follows from Lemmas 8.1 and 8.8 that φ is a monotone mapping from $(B, \preceq)$ into A. E10 follows from the definition of $\mathfrak{D}_\alpha$. E11 follows from (7.7). E12 follows from Lemma 8.11. Thus (2.9) is an admissible family. Fromula (2.10) follows from Lemma 8.8. Consequently, E has the foundation, namely (2.9). Theorem 2.6 is proved.

9. EXAMPLES.

Fig. 1 illustrates the injunction x on $R = \mathbb{Z}$ and the mold y of x. Pairs of type (r, x_r) for $r \in R$ are indicated by points and pairs of type (p, y_p) for $p \in P$ are indicated by circles. Those of pairs (p, y_p) for which $C_p \subseteq A_\alpha$, $\alpha \in Q = 2\mathbb{Z}$ are contained within the angle bounded by the dotted lines with apex in the point $(\alpha, 0)$.

In Fig. 2 a semilattice $E = E(\{B_\alpha\}_{\alpha \in \Gamma}, B, \varphi, A, \{A_\alpha\}_{\alpha \in \Gamma})$ is pictured. x and y are picked as in Fig. 1. For $p \in P$

C_p is a segment of $\mathbb{R}$ containing p. An initial covering coincides with a fundamental covering, i.e. $\Lambda = \emptyset$, $\Gamma = Q$. For $\alpha \in \Gamma$ $\mathfrak{D}_\alpha = \mathbb{Q} \cap A_\alpha$ ($\mathbb{Q}$ is the chain of rational numbers). B_α is a copy of $\mathfrak{D}_\alpha$ and $\varphi|B_\alpha$ is the corresponding isomorphism.

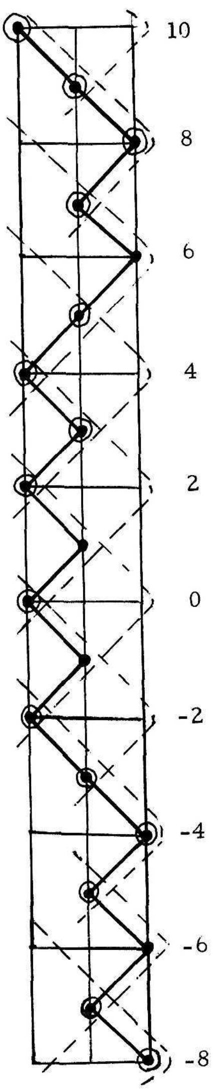

Fig. 1.

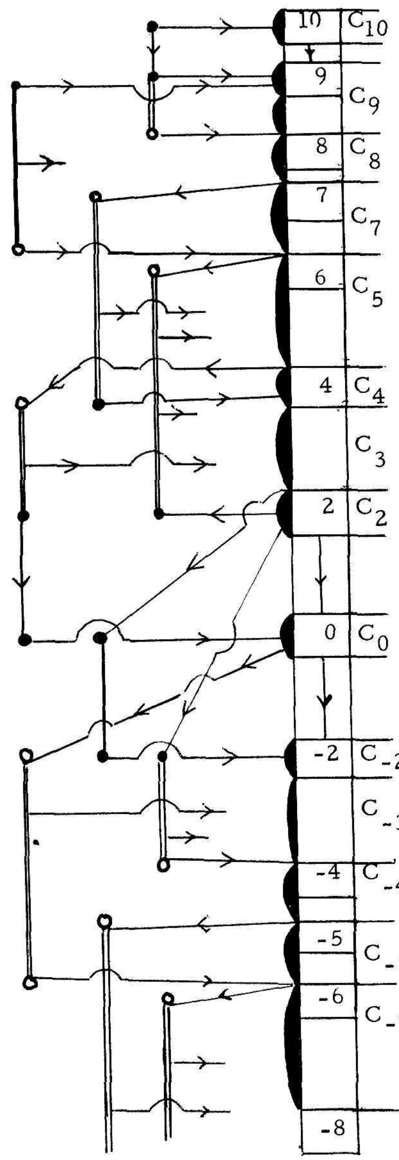

Fig. 2.

In Fig. 3 a countable semilattice $E = E(\{B_\alpha\}_{\alpha\in\Gamma}, B, \varphi, A, \{A_\alpha\}_{\alpha\in\Gamma})$ is pictured. Here $Q = \{0\} = P = R$,

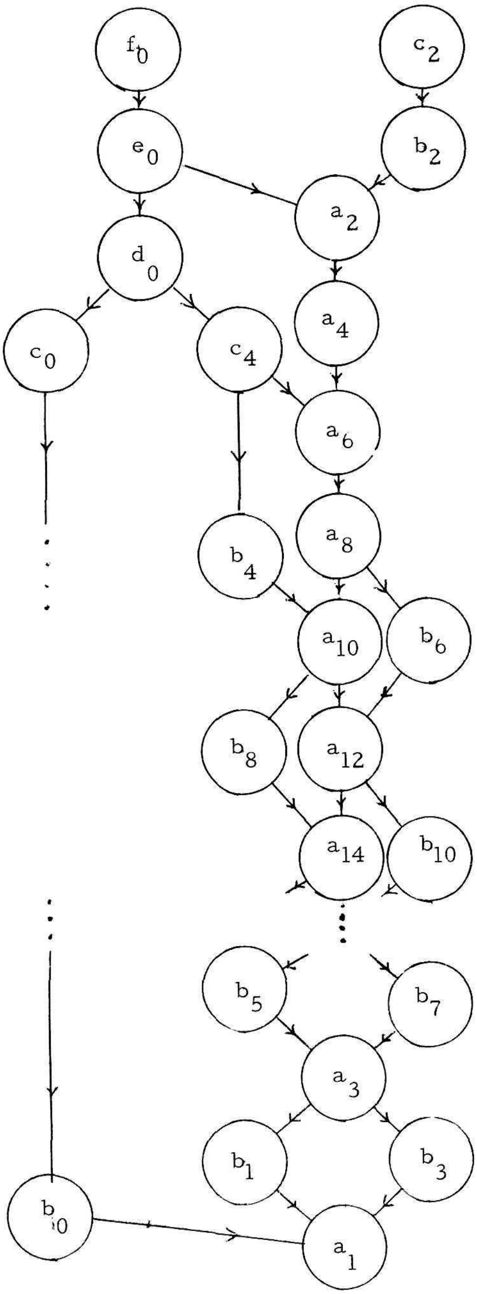

Fig. 3.

$\Lambda = \mathbb{N}$, A is of the ordinal type $\omega + \omega^*$,

$A_0 = A = \{a_k\}_{k\in\mathbb{N}}$, $A_2 = \{a_2\}$,

$A_4 = \{a_2, a_4, a_6, a_8, a_{10}\}$,

$A_{2k} = \{a_{2k+4}, a_{2k+6}\}$ for $k \geqq 3$,

$A_{4k-1} = A_{4k-3} = \{a_{2k-1}\}$ for $k \in \mathbb{N}$,

$B_0 = \{b_0, c_0, d_0, e_0, f_0\}$,

$B_2 = \{c_2, b_2\}$

$B_4 = \{b_4, c_4, d_0, e_0, f_0\}$,

$B_k = \{b_k\}$ for $k \notin \{0, 2, 4\}$.

The diagrams of all indecomposable single-trunked semilattices of width 2 and 3 and cardinality ≦ 7 are pictured below. Numbers a.b indicate cardinality and width of the subsequent semilattices. Numbers within the diagrams are a numeration of the semilattices.