ISSN 0012-2661, Differential Equations, 2007, Vol. 43, No. 10, pp. 1472–1477. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © F.E. Lomovtsev, 2007, published in Differentsial'nye Uravneniya, 2007, Vol. 43, No. 10, pp. 1433–1436.

SHORT COMMUNICATIONS

Cauchy Problem for Quasihyperbolic Factorized Differential Equations with Variable Domains of Discontinuous Operators

F. E. Lomovtsev
Belarus State University, Minsk, Belarus

Received April 5, 2001

DOI: 10.1134/S0012266107100175

Quasuhyperbolic factorized differential-operator equations with variable domains of smooth operator coefficients were considered in [1]. In the case of discontinuous operator coefficients, only hyperbolic second-order differential-operator equations were investigated [2, 3]. In the present paper, we prove the strong well-posedness and the smoothness of strong solutions of quasihyperbolic factorized differential-operator equations with variable domains of discontinuous operator coefficients; i.e., one of the main results for hyperbolic second-order differential-operator equations with discontinuous operators in [4] is generalized to the case of quasihyperbolic even-order differential-operator equations.

1. STATEMENT OF THE PROBLEMS

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. On the bounded interval $]0,T[$, consider the Cauchy problem

$$\mathscr{L}_m(t)u \equiv \left(d^2/dt^2 + A_m(t)\right)\cdots\left(d^2/dt^2 + A_1(t)\right)u = f, \qquad t \in]0,T[, \tag{1}$$

$$l_j u \equiv \left(d^j u/dt^j\right)\Big|_{t=0} = \varphi_j, \qquad 0 \le j \le 2m-1, \qquad m = 1,2,\ldots, \tag{2}$$

where u and f are functions of the variable t with values in H and the $A_k(t)$ are positive self-adjoint operators in H with t-dependent domains $D\left(A_k(t)\right)$, $t \in [0,T[$, $k = 1,\ldots,m$.

Let all operators $A_k(t)$, $k = 1,\ldots,m$, satisfy conditions I, IV, and VI in [1] for each $t \in [0,T[$. Following [1], for each $t \in [0,T[$, we equip the domains $D\left(A^{\alpha/(2m)}(t)\right)$ (dense in H) of the positive fractional powers $A^{\alpha/(2m)}(t)$ of the self-adjoint operators $A(t) = A_1^m(t)$ in H with the norms $|v|_{\alpha,t} = |A_1^{\alpha/2}(t)v|$; thus we obtain Hilbert spaces $W^\alpha(t)$, $t \in [0,T[$, $0 \le \alpha \le 2m$, with $W^0(t) = H$. In addition, we assume that the remaining conditions II, III, and V in [1] are valid only locally in the sense of the following respective conditions VII and VIII.

VII. The interval $[0,T[$ is divided into pairwise disjoint intervals $I_r = [t_r, t_{r+1}[$, $r = 0,\ldots,R$, $t_0 = 0$, $t_{R+1} = T$, so as to ensure that on each interval I_r, the inverse operators

$$A_k^{-1}(t) \in \mathscr{B}([0,T[, \mathscr{L}(H))$$

have the strong derivatives $dA_k^{-1}(t)/dt \in \mathscr{B}\left(I_r, \mathscr{L}(H)\right)$ and $d^2A_k^{-1}(t)/dt^2 \in L_\infty\left(I_r, \mathscr{L}(H)\right)$ in H such that

$$-\left(\left(dA_k^{-1}(t)/dt\right)g, g\right) \le c_k^{(1)}\left(A_k^{-1}(t)g, g\right) \qquad \forall g \in H, \qquad k = 1,\ldots,m, \tag{3}$$

$$\left|\left(\left(d^2A_k^{-1}(t)/dt^2\right)g, v\right)\right| \le c_k^{(2)}|g|\left(A_k^{-1}(t)v, v\right)^{1/2} \qquad \forall g, v \in H, \qquad k = 1,\ldots,m. \tag{4}$$

Here all constants $c_k^{(1)}$, $c_k^{(2)} \ge 0$ are independent of u, t, g, v, and r.

VIII. On each interval I_r, there exist Banach spaces V_r^{2i}, $i = 0, \dots, m$, independent of t such that $V_r^0 = H$, $D\left(\tilde{A}_k(t)\right) \subset V_r^2$, V_r^{2j} is continuously embedded in V_r^{2i} if $j > i$, $W^{2i}(t)$ is continuously embedded in V_r^{2i}, $i = 0, \dots, m$, and the strong derivatives

$$d^i \tilde{A}_k(t)/dt^i \in \mathscr{B}\left(I_r, \mathscr{L}\left(V_r^{2[j/2]+2}, V_r^{2[j/2]}\right)\right),$$
$$j = 0, \dots, 2m-2-i, \qquad i = 0, \dots, 2m-2, \qquad k = 1, \dots, m,$$

exist in H, where $[\cdot]$ is the integer part of a number.

In addition, we require that the operators $A_k(t)$, $k = 1, \dots, m$, satisfy the following matching conditions at their removable points of discontinuity t_r, $r = 1, \dots, R$.

IX. If the partition $\{I_r\}$, $r = 0, \dots, R$, of the interval $[0, T[$ consists of two or more intervals, then the following matching conditions are valid for any two adjacent intervals I_{r-1} and I_r.

(a) At the common boundary point t_r, the intersections $D\left(A_1^{3m/2}(t_r - 0)\right) \cap D\left(A_1^{3m/2}(t_r)\right)$ of the domains of the powers $A_1^{3m/2}(t_r - 0)$ and $A_1^{3m/2}(t_r)$ of the left continuations are dense in $W^{2m-1}(t_r - 0)$, $r = 1, \dots, R$, and there exists a constant $c_1 \ge 1$, independent of u and $t \in [0, T[$, such that

$$\left|A_1^{m-1/2}(t_r)\, u\right|^2 \le c_1 \left|A_1^{m-1/2}(t_r - 0)\, u\right|^2 \qquad \forall u \in W^{2m-1}(t_r - 0), \qquad r = 1, \dots, R. \tag{5}$$

(b) Inequalities (3) are valid with the sign of absolute value on the left-hand sides on I_{r-1}.

(c) On the intervals I_{r-1} and I_r, the inverse operators $A_k^{-(2m-k)/2}(t)$ of the powers $A_k^{(2m-k)/2}(t)$ have the strong derivatives $d^i A_k^{-(2m-k)/2}(t)/dt^i \in \mathscr{B}(I_p, \mathscr{L}(H))$ such that

$$A_k^{(2m-k)/2}(t)\left(d^i A_k^{-(2m-k)/2}(t)/dt^i\right) \in \mathscr{B}\left(I_p, \mathscr{L}\left(W^{i-1}(t), H\right)\right),$$
$$i = 1, 2, \qquad p = r-1, r, \qquad k = 1, \dots, m.$$

There exist strong derivatives $d^j A_k^{-1}(t)/dt^j \in \mathscr{B}(I_p, \mathscr{L}(H))$, $j = 1, \dots, 2m-2k$, $p = r-1, r$, $k = 1, \dots, m-1$, such that

$$d^j A_k^{-1}(t)/dt^j \in \mathscr{B}\left(I_p, \mathscr{L}\left(W^{2m-k-2-s+n+j}(t), W^{2m-k-s+n}(t)\right)\right), \qquad p = r-1, r,$$

where $n = 1, \dots, i-j+1$, $j = 1, \dots, i$, $i = 1, \dots, s$, $s = 1, \dots, 2m-2k$, and $k = 1, \dots, m-1$.

2. EXISTENCE AND UNIQUENESS THEOREM

Let the Banach spaces $\mathscr{E}^{2m-1,2m-1}$ be the sets of functions $u \in \mathscr{B}([0, T[, H)$ with finite norm

$$|||u|||_m = \left\{\sup_{0<t<T} \sum_{i=0}^{2m-1} \left|\frac{d^i u(t)}{dt^i}\right|^2_{2m-1-i,t}\right\}^{1/2},$$

and let the sets $D(L_m) = \Big\{u \in D\left(\tilde{L}_m\right) : \ d^s u/dt^s \in L_2\left(I_r, W^{2m-2[(s+1)/2]}(t)\right) \cap L_2(]0, T[, H)$, $s = 0, \dots, 2m$, $r = 0, \dots, R\Big\}$ belong to the spaces $\mathscr{E}^{2m-1,2m-1}$, $m = 1, 2, \dots$, where

$$D\left(\tilde{L}_m\right) = \Bigg\{u \in L_2(]0, T[, H) : \ \frac{d^s u}{dt^s} \in L_2\left(I_r, V_r^{2m-2[(s+1)/2]}\right), \ s = 0, \dots, 2m, \ r = 0, \dots, R;$$
$$\frac{d^{2m-2}u}{dt^{2m-2}}, \ \prod_{i=1}^{p}\left(\frac{d^{\alpha_i}\tilde{A}_{k_i}(t)}{dt^{\alpha_i}}\right)\left(\frac{d^{2m-2p-2-|\alpha(p)|}u}{dt^{2m-2p-2-|\alpha(p)|}}\right) \in L_2\left(I_r, W^2(t)\right),$$
$$|\alpha(p)| = 0, \dots, 2m-2p-2, \ p = 1, \dots, m-1,$$
$$1 \le k_1, \ \dots, \ k_p \le m, \ k_i \ne k_j, \ r = 0, \dots, R\Bigg\},$$

$[\cdot]$ is the integer part of a number, $\alpha(p) = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p$, and $|\alpha(p)| = \alpha_1 + \cdots + \alpha_p$. In what follows, the derivation of a priori estimates for strong solutions of the Cauchy problem (1), (2) also provides the proof of the inclusions $D(L_m) \subset \mathscr{E}^{2m-1,2m-1}$. For the spaces of strong solutions of the Cauchy problems (1), (2), we take the Banach spaces E^m that are the closures of the sets $D(L_m)$ in the norms $|||\cdot|||_m$. For the spaces of right-hand sides of Eq. (1) and the initial conditions (2), we take the Hilbert spaces $F^m = L_2(]0,T[,H) \times W^{2m-1}(0) \times \cdots \times H$ that are the sets of all elements $\mathscr{F} = \{f, \varphi_0, \ldots, \varphi_{2m-1}\} \in F^m$ with finite Hermitian norms

$$\langle\|\mathscr{F}\|\rangle_m = \left\{ \int\limits_0^T |f(t)|^2 dt + \sum_{j=0}^{2m-1} |\varphi_j|^2_{2m-1-j,0} \right\}^{1/2}.$$

The Cauchy problems (1), (2) correspond to linear unbounded operators

$$L_m \equiv \{\mathscr{L}_m(t), l_0, \ldots, l_{2m-1}\} : E^m \supset D(L_m) \to F^m$$

with dense domains $D(L_m)$, $m = 1, 2, \ldots$ If conditions I and IV in [1] and conditions VII and VIII are valid and $D(L_m) \subset \mathscr{E}^{2m-1,2m-1}$, then the operators L_m admit strong closures

$$\bar{L}_m \equiv \left\{\bar{\mathscr{L}}_m(t), l_0, \ldots, l_{2m-1}\right\} : E^m \supset D\left(\bar{L}_m\right) \to F^m.$$

The solutions of the operator equations $\bar{L}_m u = \mathscr{F}$, $\mathscr{F} \in F^m$, $m = 1, 2, \ldots$, are referred to as strong solutions of the Cauchy problems (1), (2).

Theorem 1. *If conditions* I , IV, *and* VI *in* [1] *and conditions* VII–IX *are satisfied and the strong derivative of the inverse operators* $A^{-1}(t) \in \mathscr{B}([0,T[,\mathscr{L}(H))$ *of* $A(t)$ *satisfies the inclusion* $dA^{-1}(t)/dt \in \mathscr{B}\left(I_r, \mathscr{L}\left(H, W^{2m-1}(t)\right)\right)$, $r = 0, \ldots, R$, *for* $m > 1$ *on each interval* I_r, *then, for arbitrary* $f \in L_2(]0,T[,H)$ *and* $\varphi_j \in W^{2m-1-j}(0)$, $j = 0, \ldots, 2m-1$, *there exists a unique strong solution* $u \in E^m$ *of the Cauchy problems* (1), (2), *and*

$$|||u|||_m^2 \le c_0(m)\langle\|\mathscr{F}\|\rangle_m^2, \qquad \mathscr{F} = \{f, \varphi_0, \ldots, \varphi_{2m-1}\}, \qquad c_0(m) > 0, \qquad m = 1, 2, \ldots \tag{6}$$

Proof. By virtue of conditions I, IV, and VI in [1], inequality (3), and Theorem 1 in [1], we have the estimate

$$\sup_{t\in I_r} \sum_{i=0}^{2m-1} \left|\frac{d^i u(t)}{dt^i}\right|^2_{2m-1-i,t} \le c_2 e^{c_3(t_{r+1}-t_r)} \left(\int\limits_{t_r}^{t_{r+1}} \left|\bar{\mathscr{L}}_m(t)u\right|^2 dt + \sum_{j=0}^{2m-1} |l_{j,r}u|^2_{2m-1-j,t_r} \right) \qquad (O_{t_r,t_{r+1}})$$

on each interval I_r for arbitrary $u \in D\left(\bar{L}_{m,r}\right)$, where $l_{j,r}u = \left.(d^j u/dt^j)\right|_{t=t_r}$, the domains $D\left(\bar{L}_{m,r}\right)$ are obtained from the domains $D\left(\bar{L}_m\right)$ by the replacement of $[0,T[$ by I_r, $r = 0, \ldots, R$, and $c_2, c_3 > 0$ are constants independent of u and $t \in [0,T[$. By virtue of the embeddings

$$W^{2m-1-j}(t_r - 0) \subset W^{2m-1-j}(t_r + 0), \qquad j = 0, \ldots, 2m-1, \qquad r = 1, \ldots, R,$$

which follow from (5) in view of the well-known Heinz inequality, conditions I, IV, and VI in [1], inequalities (3) and (4), and Theorem 2 in [1], for any $f_r \in L_2(I_r, H)$ and $\varphi_{j,r} = \left(d^j u_{r-1}(t_r)/dt^j\right) \in W^{2m-1-j}(t_r)$, $\varphi_{j,0} = \varphi_j$, $j = 0, \ldots, 2m-1$, there exist (recursively with respect to r) unique strong solutions $u_r \in E_r^m$ of the considered Cauchy problems on I_r, that is, solutions of the operator equations $\bar{L}_{m,r}u = \mathscr{F}_r$, $\mathscr{F}_r = \{f_r, \varphi_{0,r}, \ldots, \varphi_{2m-1,r}\} \in F_r^m$, $r = 0, \ldots, R$, where the norms of the Banach spaces E_r^m and the Hilbert spaces F_r^m are given by the left- and right-hand sides, respectively, of inequalities $\left(O_{t_r,t_{r+1}}\right)$, and $\bar{L}_{m,r} \equiv \left\{\bar{\mathscr{L}}_m(t), l_{0,r}, \ldots, l_{2m-1,r}\right\}$.

Therefore, we add inequalities (O_{t_0,t_1}) and (O_{t_1,t_2}) and estimate the right-hand sides in (O_{t_1,t_2}) first with the use of the inequalities $|v|^2_{2m-1-j,t_r+0} \le c_1|v|^2_{2m-1-j,t_r-0}$ for all $v \in W^{2m}(t_r - 0)$, $r = 1, \ldots, R$, which follow from (5), and then by the right-hand sides of inequality (O_{t_0,t_1}) (see [2]);

after that, for each $f \in L_2(]0,T[,H)$ and $\varphi_j \in W^{2m-1-j}(0)$, $j = 0,\dots,2m-1$, we find the unique function $u_{0,1} \in E^m_{0,1}$ equal to u_r on I_r, $r = 0,1$, and satisfying the equations $\bar{L}_{m,0}u = \mathscr{F}_0$ and $\bar{L}_{m,1}u = \mathscr{F}_1$ and the estimate (O_{t_0,t_2}) with the constant $c_2\left(c_1c_2+1\right)$ instead of c_2. By definition, the function $u_{0,1}$ is a strong solution of the Cauchy problem (1), (2) on the interval $[t_0,t_2[$, i.e., a solution of the equations $\bar{L}_{m,0,1}u = \mathscr{F}_{0,1}$ and $\mathscr{F}_{0,1} = \{f,\varphi_0,\dots,\varphi_{2m-1}\} \in F^m_{0,1}$, where $\bar{L}_{m,0,1} \equiv \left\{\bar{\mathscr{L}}_m(t), l_0,\dots,l_{2m-1}\right\}$, if there exists a sequence $u^{(n)}_{0,1} \in D\left(L_{m,0,1}\right)$ such that $u^{(n)}_{0,1} \to u_{0,1}$ in $E^m_{0,1}$ and $L_{m,0,1}u^{(n)}_{0,1} \to \mathscr{F}_{0,1}$ in $F^m_{0,1}$ as $n \to \infty$. The norms of the Banach spaces $E^m_{0,1}$ and the Hilbert spaces $F^m_{0,1}$ are determined by the left- and right-hand sides of inequality (O_{t_0,t_2}), respectively, and the domains $D\left(L_{m,0,1}\right)$ are obtained from the domain $D\left(L_m\right)$ by the replacement of $[0,T[$ by $[t_0,t_2[$. By using condition IX (b), one can derive the estimate

$$\sup_{t_0<t<t_1}\sum_{i=0}^{2m-1}\left|\frac{d^iu(t)}{dt^i}\right|^2_{2m-1-j,t} \le c_2e^{c_3(t_1-t_0)}\left(\int_{t_0}^{t_1}\left|\bar{\mathscr{L}}_m(t)u\right|^2dt + \sum_{j=0}^{2m-1}\left|\frac{d^ju\left(t_1\right)}{dt^j}\right|^2_{2m-1-j,t_1-0}\right) \tag{7}$$

for arbitrary $u \in D\left(\bar{L}_{m,0}\right)$. Obviously, the function u_0 is the unique strong solution of the Cauchy problems (1), (2) in reverse time on I_0 for $f_0 = f \in L_2\left(I_0,H\right)$ and $\varphi_{j,1} \in W^{2m-1-j}\left(t_1-0\right)$, $j = 0,\dots,2m-1$. Since $L_2\left(I_0,W^m(t)\right)$ is dense in $L_2\left(I_0,H\right)$, it follows that there exists a sequence $f^{(n)}_0 \in L_2\left(I_0,W^m(t)\right)$ converging to $f_0 = f$ in $L_2\left(I_0,H\right)$, and, by virtue of condition IX (a), there exists a sequence $\varphi^{(n)}_{j,1} \in D\left(A_1^{3m/2}\left(t_1-0\right)\right) \cap D\left(A_1^{3m/2}\left(t_1+0\right)\right)$ converging to $\varphi_{j,1}$ in $W^{2m-1-j}\left(t_1-0\right)$, $j = 0,\dots,2m-1$, as $n \to \infty$.

Theorem 2. *Let the assumptions of Theorem* 1 *with condition* IX *replaced by a condition* IX (c) *on the interval I_r be valid; then for any $f_r \in \mathscr{H}^m_r$ and $\varphi_{j,r} \in W^{3m-1-j}\left(t_r+0\right)$, $j = 0,\dots,2m-1$, the Cauchy problems* (1), (2) *on the interval I_r have a unique strong solution $u_r \in E^m_r$ such that $d^iu_r/dt^i \in \mathscr{H}^{2m-i}_r$, $i = 0,\dots,2m$, $r = 0,\dots,R$, where $\mathscr{H}^\alpha_r = L_2\left(I_r,W^\alpha(t)\right)$ are Hilbert spaces.*

By the above-proved Theorem 2 on the smoothness for $f^{(n)}_0$ and $\varphi^{(n)}_{j,1}$, strong solutions $u^{(n)}_0$ of the Cauchy problem (1), (2) in reverse time on I_0 belong to the domains $D\left(L_{m,0}\right)$ by virtue of condition IX (c).

It follows from (7) that $u^{(n)}_0 \to u_0$ in E^m_0 provided that $f^{(n)}_0 \to f_0$ in $L_2\left(I_0,H\right)$ and $\varphi^{(n)}_{j,1} \to \varphi_{j,1}$ in $W^{2m-1-j}\left(t_1-0\right)$, $j = 0,\dots,2m-1$, as $n \to \infty$. By applying Theorem 2 to the Cauchy problems (1), (2) on the interval I_1, we find that, for any $f^{(n)}_1 \in L_2\left(I_1,W^m(t)\right)$ and $\varphi^{(n)}_{j,1}$, $j = 0,\dots,2m-1$, their strong solutions $u^{(n)}_1$ belong to $D\left(L_{m,1}\right)$ by virtue of condition IX (c). It follows from inequalities (O_{t_1,t_2}) that $u^{(n)}_1 \to u_1$ in E^m_1 provided that $f^{(n)}_1 \to f_1 = f$ in $L_2\left(I_1,H\right)$ and $\varphi^{(n)}_{j,1} \to \varphi_{j,1}$ in $W^{2m-1-j}\left(t_1+0\right)$, $j = 0,\dots,2m-1$, as $n \to \infty$ by virtue of the inequality $|v|^2_{2m-1-j,t_1+0} \le c_1|v|^2_{2m-1-j,t_1-0}$ for all $v \in W^{2m}\left(t_1-0\right)$. Therefore, one can choose $u^{(n)}_{0,1} = u^{(n)}_r$ on I_r, $r = 0,1$, since the estimate (O_{t_0,t_2}) with the constant $c_2\left(c_1c_2+1\right)$ instead of c_2 implies that $u^{(n)}_{0,1} \to u_{0,1}$ in $E^m_{0,1}$, while $L_{m,0,1}u^{(n)}_{0,1} \to \mathscr{F}_{0,1}$ in $F^m_{0,1}$ as $n \to \infty$.

By using Theorems 1 and 2 in [1], in a similar way, one can find the unique strong solution $u_2 \in E^m_2$ of the Cauchy problems (1), (2) on I_2 for $f_2 = f \in L_2\left(I_2,H\right)$ and $\varphi_{j,2} \in W^{2m-1-j}\left(t_2+0\right)$, $j = 0,\dots,2m-1$. By definition, the function $u_{0,2}$ equal to u_r on the intervals I_r, $r = 0,1,2$, is a strong solution of the original Cauchy problems on the interval $[t_0,t_3[$ if there exists a sequence $u^{(n)}_{0,2} \in D\left(L_{m,0,2}\right)$ such that $u^{(n)}_{0,2} \to u_{0,2}$ in $E^m_{0,2}$ and $L_{m,0,2}u^{(n)}_{0,2} \to \mathscr{F}_{0,2} = \{f,\varphi_0,\dots,\varphi_{2m-1}\}$ in $F^m_{0,2}$ as $n \to \infty$. The norms of the Banach spaces $E^m_{0,2}$ and the Hilbert spaces $F^m_{0,2}$ are defined by the left- and right-hand sides of inequality (O_{t_0,t_3}) with the constant $c_2\left(c_1c_2+1\right)^2$ instead of c_2, whose derivation is similar to that of inequality (O_{t_0,t_2}).

By repeating the above-performed considerations for the intervals I_1 and I_2 instead of the intervals I_0 and I_1, we construct a sequence $u^{(n)}_{1,2} \in D\left(L_{m,1,2}\right)$ such that $u^{(n)}_{1,2} \to u_{1,2}$ in $E^m_{1,2}$, $\mathscr{L}_m(t)u^{(n)}_{1,2} \to f$

in $L_2(]t_1,t_3[\,,H)$, and $d^j u_{1,2}^{(n)}(t_1)/dt^j \to d^j u_1(t_1)/dt^j$ in $W^{2m-1-j}(t_1+0)$, $j=0,\ldots,2m-1$, as $n\to\infty$. Then the sequence $u_{0,2}^{(n)} = q_n^- u_{0,1}^{(n)} + q_n^+ u_{1,2}^{(n)}$ belongs to the domains $D(L_{m,0,2})$ for all sufficiently large n, where, in the partition of unity $q_n^-(t)+q_n^+(t)=1$ for all $t\in\mathbb{R}$, we have $q_n^+(t) = n\int_{\mathbb{R}} w(n(t-s))\mathscr{X}(s)ds$ for some function $w\in C^\infty(\mathbb{R})$ such that $w\ge 0$, $w(t)=0$ for $|t|>1$, and $\int_{\mathbb{R}} w(t)dt = 1$ and for the characteristic function $\mathscr{X}(t)$ of the interval $[\bar t,+\infty[$, $\bar t = t_1 + (t_2-t_1)/2$. It is known that $q_n^+(t)\in C^\infty(\mathbb{R})$, $0\le q_n^+(t)\le 1$ for all $t\in\mathbb{R}$, $q_n^+(t)=0$ for $t\le \bar t - 1/n$, and $q_n^+(t)=1$ for $t\ge \bar t+1/n$.

We have the inequalities

$$\int_{t_0}^{t_3}\left|\mathscr{L}_m(t)u_{0,2}^{(n)} - f\right|^2 dt \le 3\int_{t_0}^{t_2}\left|\mathscr{L}_m(t)u_{0,1}^{(n)} - f\right|^2 dt + 3\int_{t_1}^{t_3}\left|\mathscr{L}_m(t)u_{1,2}^{(n)} - f\right|^2 dt$$
$$+\, c_4\sum_{i=0}^{2m-1}\sum_{k=1}^{2m-i}\int_{\bar t-1/n}^{\bar t+1/n}\left|\frac{d^k q_n^-(t)}{dt^k}\frac{d^i u_{0,1}^{(n)}}{dt^i} + \frac{d^k q_n^+(t)}{dt^k}\frac{d^i u_{1,2}^{(n)}}{dt^i}\right|^2_{2m-1-i,t} dt \qquad (8)$$

for all sufficiently large n. By using the estimates for the derivatives, $\left|d^k q_n^+(t)/dt^k\right|\le c_5 n^k$, $t\in\mathbb{R}$, $k=1,\ldots,2m$, one can estimate the last term in (8) from above via

$$c_6 n^{2m-1}\left(\left\|u_{0,1}^{(n)} - u_{0,2}\right\|^2_{E^m_{0,1}} + \left\|u_{1,2}^{(n)} - u_{0,2}\right\|^2_{E^m_{1,2}}\right),$$

where $\|\cdot\|_{E^m_{0,1}}$ and $\|\cdot\|_{E^m_{1,2}}$ are the norms of the Banach spaces $E^m_{0,1}$ and $E^m_{1,2}$. Here $c_i\ge 0$, $i=4,5,6$, are constants independent of n. Then it follows from (8) that $\mathscr{L}_m(t)u_{0,2}^{(n)}\to f$ in $L_2(]t_0,t_3[\,,H)$ as $n\to\infty$ since, by definition, $\mathscr{L}_m(t)u_{0,1}^{(n)}\to f$ in $L_2(]t_0,t_2[\,,H)$ and $\mathscr{L}_m(t)u_{1,2}^{(n)}\to f$ in $L_2(]t_1,t_3[\,,H)$ as $n\to\infty$, and $u_{0,1}^{(n)}$ and $u_{1,2}^{(n)}$ can be chosen so as to ensure that

$$\left\|u_{0,1}^{(n)} - u_{0,2}\right\|^2_{E^m_{0,1}} \quad\text{and}\quad \left\|u_{1,2}^{(n)} - u_{0,2}\right\|^2_{E^m_{1,2}} \le 1/2^n, \qquad n=1,2,\ldots$$

Then the estimate (O_{t_0,t_3}) with the constant $c_2(c_1c_2+1)^2$ instead of c_2 implies that $u_{0,2}^{(n)}\to u_{0,2}$ in $E^m_{0,2}$ as $n\to\infty$, and so on. As a result, for given f and φ_j, $j=0,\ldots,2m-1$, we obtain the sewed unique strong solution $u\in E^m$ of the Cauchy problems (1), (2), which is equal to u_r on the intervals I_r, $r=0,\ldots,R$, and satisfies inequalities (6) for $c_0(m) = c_2(c_1c_2+1)^R\exp(c_3T)$.

Remark. If $B_k(t)\in L_\infty\left(]0,T[\,,\mathscr{L}\left(W^{2m-1-k}(t),H\right)\right)$, $k=0,\ldots,2m-1$, then, by using continuation with respect to the parameter, the assertion of Theorem 1 [with larger values of $c_0(m)$ is necessary] can be generalized to the equations

$$\mathscr{L}_m(t)u + \sum_{k=0}^{2m-1} B_k(t)d^k u/dt^k = f, \qquad t\in\,]0,T[, \qquad m=1,2,\ldots$$

3. EXAMPLE OF MIXED PROBLEMS

In the bounded domain $G=\,]0,T[\,\times\Omega$, $\Omega\subset\mathbb{R}^n$, $n\ge 1$, of the variables t and $x=(x_1,\ldots,x_n)$ with a sufficiently smooth lateral surface $\Gamma = [0,T[\,\times S$, we prove the well-posedness of mixed problems for the hyperbolic partial differential equations

$$\prod_{k=1}^{m}\left(\frac{\partial^2}{\partial t^2} + \sum_{r=0}^{p(t)} a_{k,r}(t)(-\Delta)^r\right.$$
$$\left. +\, b_{k,0}(t)\frac{\partial}{\partial t} + \sum_{r=0}^{[p(t)/2]} b_{k,1,r}(t)(-\Delta)^r\right)u(t,x) = f(t,x), \qquad (t,x)\in G,$$

where the coefficients $a_{k,r}$ and $b_{k,0}$, $b_{k,1,r}$ are $\max\{2m-2,2\}$ and $2m-2k$ times, respectively, piecewise continuously differentiable functions of the variable $t \in [0,T[$ with finitely many nonsmoothness points and jump discontinuities, $a_{k,p(t)}$ are distinct for each $t \in [0,T[$, $a_{k,p(t)} > 0$, and $p(t) \geq 0$ is an integer-valued nonincreasing function of the variable $t \in [0,T[$ with finitely many discontinuity points, under the boundary conditions

$$\Delta^i u(t,x)\big|_\Gamma = 0, \qquad \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2, \qquad i = 0,\ldots,mp(t)-1, \qquad t \in [0,T[,$$

and the initial conditions $\partial^j u(0,x)/\partial t^j = \varphi_j(x)$, $x \in \Omega$, $j = 0,\ldots,2m-1$, $m = 1,2,\ldots$

REFERENCES

1. Lomovtsev, F.E., *Differ. Uravn.*, 2007, vol. 43, no. 6, pp. 795–812.
2. Lomovtsev, F.E., *Differ. Uravn.*, 1997, vol. 33, no. 10, pp. 1394–1403.
3. Lomovtsev, F.E., *Dokl. Nats. Akad. Navuk Belarusi*, 2001, vol. 45, no. 3, pp. 37–40.
4. Lomovtsev, F.E., Boundary Value Problems for Differential-Operator Equations with Variable Domains of Smooth and Discontinuous Operator Coefficients, *Doctoral (Phys.–Math.) Dissertation*, Minsk, 2003.