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PARTIAL DIFFERENTIAL EQUATIONS

Euler–Poisson–Darboux Differential-Operator Equation with Variable Domains of Smooth Operators

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Abstract—We prove the strong well-posed solvability of the Cauchy problem for a second-order singular hyperbolic differential equation with variable domain of variable unbounded operator coefficients and for the mixed problem for a complete equation of string vibrations with a strong singularity in time and with a time-dependent boundary condition.

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INTRODUCTION

Singular second-order hyperbolic differential-operator equations with constant domains of variable unbounded operators were considered in [1]. The first nonsingular hyperbolic differential-operator equations with variable domains were studied in [2, 3]. The paper [4] dealt with the existence, uniqueness, continuous dependence, and traces of strong solutions of two mixed problems for the equation of string vibrations with time singularities of a second-order differential operator in another variable. In the present paper, we use the method of energy inequalities to prove the existence and uniqueness of strong solutions of the Cauchy problem for a singular hyperbolic differential-operator equation of the Euler–Poisson–Darboux type with variable domains of variable unbounded operators and the continuous dependence of these solutions on the right-hand side of the equation. Their uniqueness and continuous dependence follow from the energy inequality derived on the basis of abstract smoothing operators, and their existence is a consequence of the density of the range, which is proved with the use of Lemma 4.2 (see below) about the adjoint operator. These results are applied to the mixed problem for a singular complete equation of string vibrations under a nonstationary boundary condition.

1. STATEMENT OF THE CAUCHY PROBLEM

On a bounded interval $]0,T[$, we consider the singular differential-operator equation

$$\frac{d^2u(t)}{dt^2} + \frac{B(t)}{t}\frac{du(t)}{dt} + A(t)u(t) = f(t), \qquad t \in]0,T[, \tag{1.1}$$

with the homogeneous initial conditions

$$u|_{t=0} = 0, \qquad \left.\frac{du(t)}{dt}\right|_{t=0} = 0, \tag{1.2}$$

where u and f are abstract functions of the variable t ranging in a Hilbert space H and $A(t)$ and $B(t)$ are linear unbounded operators in H with t-dependent domains $D(A(t))$ and $D(B(t))$, respectively. Let the operators $A(t)$ and $B(t)$ satisfy the following conditions.

(A_0). For each $t \in [0,T]$, the operators $A(t)$ are self-adjoint in H, and

$$(A(t)u,u) \geq a_0|u|^2 \qquad \forall u \in D(A(t)), \qquad a_0 > 0. \tag{1.3}$$

(A_1). For all $t \in [0,T]$, their inverse operators $A^{-1}(t) \in \mathcal{B}([0,T],\mathcal{L}(H))$ are strongly continuous with respect to t in H and have a bounded strong derivative $dA^{-1}(t)/dt \in \mathcal{B}([0,T],\mathcal{L}(H))$ in H such that

$$-((dA^{-1}(t)/dt)g,g) \le a_1(A^{-1}(t)g,g) \qquad \forall g \in H, \qquad a_1 \ge 0. \tag{1.4}$$

(A_2). For all $t \in]0,T[$, the operators $dA^{-1}(t)/dt$ have a bounded strong derivative $d^2A^{-1}(t)/dt^2 \in L_\infty(]0,T[,\mathcal{L}(H))$ in H such that

$$|((d^2A^{-1}(t)/dt^2)g,v)| \le a_2|g|(A^{-1}(t)v,v)^{1/2} \qquad \forall g,v \in H, \qquad a_2 \ge 0. \tag{1.5}$$

(B_0). For each $t \in [0,T]$, the operators $B(t)$ are subordinate to the square root $A^{1/2}(t)$ of the operators $A(t)$ (i.e., $|B(t)u| \le a_3|A^{1/2}(t)u|$ for all $u \in D(A^{1/2}(t)$, $a_3 > 0)$, and the following estimate holds:

$$-\operatorname{Re}(B(t)u,u) \le b_0t|u|^2 \qquad \forall u \in D(A(t)), \qquad b_0 \ge 0. \tag{1.6}$$

(B_1). For all $t \in [0,T]$, the operators $B(t)(dA^{-1}(t)/dt) \in \mathcal{B}([0,T],\mathcal{L}(H))$ are bounded, and

$$-\operatorname{Re}(B(t)u,A(t)u) \le b_1t(A(t)u,u) \qquad \forall u \in D(A(t)), \qquad b_1 \ge 0, \tag{1.7}$$

$$|(B(t)(dA^{-1}(t)/dt)g,h)| \le b_2t|g|(A^{-1}(t)h,h)^{1/2} \qquad \forall g,h \in H, \qquad b_2 \ge 0. \tag{1.8}$$

Here $\mathcal{B}([0,T],\mathcal{L}(H))$ and $L_\infty(]0,T[,\mathcal{L}(H))$ are the Banach spaces of all operator-valued functions bounded for all and almost all $t \in [0,T]$, respectively, and ranging in the space of linear continuous operators acting in H.

We encounter the problem as to whether the Cauchy problem (1.1), (1.2) is well posed in the Hadamard sense in the class of strong (generalized) solutions.

Remark 1.1. Unlike the second initial condition in (1.2), the first one is not necessarily homogeneous (see [1]). In [1], the sets $D(A(t))$ and $D(B(t))$ are independent of t; therefore, inequalities (1.4) and (1.5) are absent (they are necessarily true), the right-hand sides of inequalities (1.6) and (1.7) do not contain the factor t, and inequality (1.8) without the multiplier t is necessarily true. In papers earlier than [1], the operators $B(t)$ were assumed to be bounded.

2. DEFINITION OF STRONG SOLUTIONS OF THE CAUCHY PROBLEM

Let us introduce spaces in which the operator of the posed Cauchy problem acts. Let the space of strong solutions of the Cauchy problem (1.1), (1.2) be the Hilbert space E obtained as the closure of the set

$$\begin{aligned} D(L(t)) = \{u \in \mathcal{H} = L_2(]0,T[,H):\ & u(t) \in D(A(t)),\ du(t)/dt \in D(B(t)),\ t \in [0,T];\\ & du/dt,\ d^2u/dt^2,\ (B(t)/t)(du/dt),\ A(t)u \in \mathcal{H};\ u(0) = du(0)/dt = 0\} \end{aligned}$$

of all smooth solutions in the Hermitian norm

$$\|u\|_E = \left[\int_0^T (|du/dt|^2 + |A^{1/2}(t)u|^2)dt\right]^{1/2}. \tag{2.1}$$

Let the space of right-hand sides f of Eq. (1.1) be the Banach space F that is the closure of the set $\mathcal{H}$ in the norm

$$\|f\|_F = \sup_{v\in\mathcal{H}}\left\{\left|\int_0^T (T-t)(f,v)\,dt\right|\bigg/\|v\|_0\right\}, \qquad \|v\|_0 = \left(\int_0^T |v(t)|^2\,dt\right)^{1/2}. \tag{2.2}$$

Obviously, the negative space F is the set of all antilinear continuous functionals on the positive Hilbert space E_1 with Hermitian norm $\|w\|_{E_1} = \left[\int_0^T (T-t)^{-2}|w(t)|^2\,dt\right]^{1/2}$; i.e., $F = (E_1)'$ is the strong dual space of E_1.

To the Cauchy problem (1.1), (1.2), there corresponds a linear unbounded operator $L(t):\ E \supset D(L(t)) \to F$ with dense domain $D(L(t))$. The following assertion can be proved in a standard way.

Lemma 2.1. *If the operators $A(t)$ satisfy condition (A_0) and the set*

$$\widetilde{D}(L(t)) = \{v \in D(L(t)) : \ v(t) \in D(B^*(t)), \ t \in [0,T]; \ (B^*(t)/t)v \in \mathcal{H}\}$$

is dense in $\mathcal{H}$, where $B^*(t) : H \supset D(B^*(t)) \to H$ are the adjoint operators of $B(t) : \ H \supset D(B(t)) \to H$ in H, then the operator $L(t)$ is closable.

Let $\overline{L}(t) : \ E \supset D(\overline{L}(t)) \to F$ be the strong closure of the operator $L(t)$. By the definition of the strong closure $\overline{L}(t)$, a function $u \in E$ belongs to the domain $D(\overline{L}(t))$ of the operator $\overline{L}(t)$ if there exists a sequence $u_n \in D(L(t))$ and a functional $f \in F$ such that $\|u_n - u\|_E \to 0$ and $\|L(t)u_n - f\|_F \to 0$ as $n \to \infty$. In this case, we set $\overline{L}(t)u = \lim_{n\to\infty} L(t)u_n = f$.

Definition. The solutions $u \in D(\overline{L}(t))$ of the operator equation $\overline{L}(t)u = f$, $f \in F$, are referred to as the strong solutions of the Cauchy problem (1.1), (1.2).

3. ENERGY INEQUALITY OF THE CAUCHY PROBLEM

Let us derive an energy inequality for the strong solutions of the Cauchy problem (1.1), (1.2).

Theorem 3.1. *If conditions (A_0), (A_1), and (B_0) are satisfied and the set $\widetilde{D}(L(t))$ is dense in $\mathcal{H}$, then*

$$\|u\|_E \le c_1 \|\overline{L}(t)u\|_F \qquad \forall u \in D(\overline{L}(t)), \qquad c_1 = 2\exp(\max\{a_1, 2b_0\}T). \tag{3.1}$$

Proof. In the Hilbert space H, we introduce the set of abstract smoothing operators $A_\varepsilon^{-1}(t) = (I + \varepsilon A(t))^{-1}$, $\varepsilon > 0$, ranging in $D(A(t))$. They have the following properties [2]:

(a) the operator norm satisfies the inequality $\|A_\varepsilon^{-1}(t)\|_{\mathcal{L}(\mathcal{H})} \le 1$ for all $\varepsilon > 0$ and for all $t \in [0,T]$;

(b) $A_\varepsilon^{-1}(t)g \to g$ in H uniformly with respect to $t \in [0,T]$ as $\varepsilon \to 0$ for each $g \in H$;

(c) for all $t \in [0,T]$ in H, there exists a strong derivative $dA_\varepsilon^{-1}(t)/dt \in L_\infty(]0,T[, \mathcal{L}(H))$.

In the derivation of inequality (3.1), the unbounded and nondifferentiable operators $A(t)$ are approximated by the bounded operators $A(t)A_\varepsilon^{-1}(t)$, $\varepsilon > 0$, which are strongly differentiable with respect to t in H for all $t \in [0,T]$:

$$\frac{d(A(t)A_\varepsilon^{-1}(t))}{dt} = -\frac{1}{\varepsilon}\frac{dA_\varepsilon^{-1}(t)}{dt} = -A(t)A_\varepsilon^{-1}(t)\frac{dA^{-1}(t)}{dt}A(t)A_\varepsilon^{-1}(t), \qquad \varepsilon > 0. \tag{3.2}$$

By using the above-mentioned properties of the operators $A_\varepsilon^{-1}(t)$, formula (3.2), and the estimate (1.4) and by following [2], one can prove the inequality

$$\int_0^T e^{c(T-t)}\left|\frac{du}{dt}\right|^2 dt + \int_0^T e^{c(T-t)}|A^{1/2}(t)u|^2\,dt \le 2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(\frac{d^2u}{dt^2} + A(t)u, \frac{du}{dt}\right)dt$$
$$- c\int_0^T e^{c(T-t)}(T-t)\left|\frac{du}{dt}\right|^2 dt + (a_1 - c)\int_0^T e^{c(T-t)}(T-t)|A^{1/2}(t)u|^2\,dt \qquad \forall c \ge 0$$

for all $u \in D(L(t))$.

On the right-hand side of this inequality, we add and subtract the expression

$$\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)((B(t)/t)(du/dt), (du/dt))\,dt;$$

then for any constant $c \geq a_1$, we obtain the inequality

$$\int_0^T e^{c(T-t)}(T-t)\left(\left|\frac{du}{dt}\right|^2 + |A^{1/2}(t)u|^2\right) dt \leq 2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(L(t)u, \frac{du}{dt}\right) dt$$
$$- 2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(\frac{B(t)}{t}\frac{du}{dt}, \frac{du}{dt}\right) dt - c\int_0^T e^{c(T-t)}(T-t)\left|\frac{du}{dt}\right|^2 dt \quad \forall u \in D(L(t)). \tag{3.3}$$

By virtue of the estimate (1.6), the right-hand side of inequality (3.3) does not exceed the quantity

$$2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(L(t)u, \frac{du}{dt}\right) dt + (2b_0 - c)\int_0^T e^{c(T-t)}(T-t)\left|\frac{du}{dt}\right|^2 dt,$$

whose half for $c = c_2 = \max\{a_1, 2b_0\}$ and for the norm (2.2) can be estimated from above by the quantity

$$\frac{\left|\int_0^T (T-t)\left(L(t)u, e^{c_2(T-t)}du/dt\right) dt\right|}{\|du/dt\|_0}\|u\|_E = \frac{\left|\int_0^T (T-t)(L(t)u, v)\, dt\right|}{\|e^{c_2(t-T)}v\|_0}\|u\|_E \leq e^{c_2 T}\|L(t)u\|_F\|u\|_E, \tag{3.4}$$

since $\|e^{c_2(t-T)}v\|_0 \geq e^{-c_2 T}\|v\|_0$ for all $v \in \mathcal{H}$.

After the estimation of the left-hand side of inequality (3.3) from below by $\|u\|_E^2$ and its right-hand side from above by the doubled expression (3.4) and after the elimination of $\|u\|_E$, we obtain inequality (3.1), first for smooth solutions $u \in D(L(t))$ and then, after the passage to the limit, for all strong solutions $u \in D(\overline{L}(t))$. The proof of Theorem 3.1 is complete.

The energy inequality (3.1) implies the following assertion.

Corollary 3.1. *If the assumptions of Theorem* 3.1 *are satisfied*, *then* $R(\overline{L}(t)) = \overline{R(L(t))}$, *where* $R(\overline{L}(t))$ *is the range of the operator* $\overline{L}(t)$ *and* $\overline{R(L(t))}$ *is the closure of the range* $R(L(t))$ *of the operator* $L(t)$ *in* F.

This corollary implies the existence of strong solutions of the Cauchy problem (1.1), (1.2) for all f in the Banach subspace $\overline{R(L(t))}$ of the space F.

4. THE RANGE OF THE CAUCHY PROBLEM IS DENSE

For the existence of strong solutions of this Cauchy problem for all $f \in F$, we show that the set $R(L(t))$ is dense in F.

Theorem 4.1. *If conditions* (A_0)–(A_2), (B_0), *and* (B_1) *are satisfied and the set* $\widetilde{D}(L(t))$ *is dense in* $\mathcal{H}$, *then for each function* $f \in F$, *there exists a unique strong solution* $u \in E$ *of the Cauchy problem* (1.1), (1.2), *which satisfies the estimate*

$$\|u\|_E \leq c_1\|f\|_F. \tag{4.1}$$

Proof. By Corollary 4.1, to prove the everywhere strong solvability of the Cauchy problem (1.1), (1.2), it suffices to prove by contradiction that the range $R(L(t))$ is dense in F. Let $F \neq \overline{R(L(t))}$. Then, by the well-known corollary of the Hahn–Banach theorem, on the reflexive Banach space F, there exists a linear continuous functional $0 \neq v \in F' = (E_1)'' = E_1$ orthogonal to the set $\overline{R(L(t))}$. Here single and double primes stand for the first and second strong dual spaces, respectively. Since

$L(t)u \in \mathcal{H}$ for all $u \in D(L(t))$, it follows that the values of antilinear functionals $L(t)u \in F = (E_1)'$ on $v \in E_1$ are equal to

$$\int_0^T (T-t)(L(t)u, v)\, dt = 0 \qquad \forall u \in D(L(t)). \tag{4.2}$$

To prove that the set $R(L(t))$ is dense in the space F by contradiction, it suffices to show that relation (4.2) implies that $v = 0$.

By virtue of the double differentiability of the operators $A^{-1}(t)$ with respect to t for each $\tau \in [0, T[$, in relation (4.2), we set $u = A^{-1}(t)h$, where

$$h \in M_\tau = \{h \in \mathcal{H}_\tau = L_2(]\tau, T[, H): \ (1/t)(dh/dt),\ d^2h/dt^2 \in \mathcal{H}_\tau;\ h(t) = dh(t)/dt = 0,\ t \in [0, \tau]\},$$

and obtain the identity

$$\int_\tau^T \left(\frac{d^2h}{dt^2}, A^{-1}(t)(T-t)v\right) dt = -\int_\tau^T \left(\frac{d^2A^{-1}(t)}{dt^2} h, (T-t)v\right) dt$$
$$- 2\int_\tau^T \left(\frac{dA^{-1}(t)}{dt}\frac{dh}{dt}, (T-t)v\right) dt - \int_\tau^T \left(\frac{B(t)}{t}\frac{dA^{-1}(t)}{dt} h, (T-t)v\right) dt$$
$$- \int_\tau^T \left(\frac{B(t)A^{-1}(t)}{t}\frac{dh}{dt}, (T-t)v\right) dt - \int_\tau^T (h, (T-t)v)\ dt \qquad \forall h \in M_\tau. \tag{4.3}$$

Suppose that, for each $\tau \in [0, T[$, the function w is a solution of the Cauchy problem $dw/dt = te^{c(t-T)}v$, $t \in]\tau, T[$; $w(\tau) = 0$ in $\mathcal{H}_\tau$, where v is the function occurring in identity (4.2). Consequently, the function

$$v = e^{c(T-t)}t^{-1}(dw/dt)$$

belongs to the space $\mathcal{H}_\tau$, $\tau \in [0, T[$, for all $c \geq 0$, since the embedding $E_1 \subset \mathcal{H}$ holds.

Then from identity (4.3), we obtain the identity

$$\int_\tau^T \left(\frac{d}{dt}t\left(\frac{1}{t}\frac{dh}{dt}\right), A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right) dt = \int_\tau^T e^{c(T-t)}\Phi(h, w)\, dt \qquad \forall h \in M_\tau, \tag{4.4}$$

where the sesquilinear form is as follows:

$$\Phi(h, w) = -\left(\frac{d^2A^{-1}(t)}{dt^2} h, (T-t)\frac{1}{t}\frac{dw}{dt}\right) - 2\left(\frac{dA^{-1}(t)}{dt}\frac{dh}{dt}, (T-t)\frac{1}{t}\frac{dw}{dt}\right)$$
$$- \left(\frac{B(t)}{t}\frac{dA^{-1}(t)}{dt} h, (T-t)\frac{1}{t}\frac{dw}{dt}\right) - \left(\frac{B(t)A^{-1}(t)}{t}\frac{dh}{dt}, (T-t)\frac{1}{t}\frac{dw}{dt}\right)$$
$$- \left(h, (T-t)\frac{1}{t}\frac{dw}{dt}\right).$$

In the inner product in the first integral in this identity, one cannot integrate by parts with respect to t from the left to the right since, by construction, the function $t^{-1}(dw/dt)$ is not differentiable with respect to t in $\mathcal{H}_\tau$.

Since, by virtue of conditions (A_1), (A_2), and (B_1), the norms of the operators $d^{i+1}A^{-1}(t)/dt^{i+1}$ and $B(t)(d^iA^{-1}(t)/dt^i)$, $i = 0, 1$, in the space $\mathcal{L}(H)$ are bounded uniformly for almost all $t \in]0, T[$, it follows from the Cauchy–Schwarz inequality and the inequality

$$\int_\tau^T \frac{1}{t^2}|h|^2\, dt \leq 4\int_\tau^T \left|\frac{dh}{dt}\right|^2\ dt, \qquad \int_\tau^T |h|^2\, dt \leq \frac{1}{12}T^4\int_\tau^T \frac{1}{t^2}\left|\frac{dh}{dt}\right|^2\ dt$$

applied to the right-hand side of identity (4.4) that there exists a constant $c_3 > 0$ such that

$$\left|\int_\tau^T \left(\left[\left(\frac{d}{dt}\right) t\right]\left(\frac{1}{t}\frac{dh}{dt}\right), A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right) dt\right|$$
$$\leq c_3\left(\int_\tau^T \frac{1}{t^2}\left|\frac{dw}{dt}\right|^2 dt\right)^{1/2}\left(\int_\tau^T \frac{1}{t^2}\left|\frac{dh}{dt}\right|^2 dt\right)^{1/2} \qquad \forall h \in M_\tau. \tag{4.5}$$

This inequality implies that the linear functional

$$\frac{1}{t}\frac{dh}{dt} \to \int_\tau^T \left(\left(\frac{d}{dt}t\right)\left[\frac{1}{t}\frac{dh}{dt}\right], A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right) dt$$

is continuous in $\mathcal{H}_\tau$. Consequently, the function $A^{-1}(t)(T-t)e^{c(T-t)}t^{-1}(dw/dt)$ belongs to the domain of the adjoint operator of the operator generated by the differential expression $d(tg(t))/dt$ on functions $g(t) = t^{-1}(dh/dt)$, $h \in M_\tau$.

For the simultaneous integration with respect to the variable t and multiplication by t on the left-hand side in identity (4.4), we need the following assertion.

Lemma 4.1 [2]. *Let X, Y, and Z be Banach spaces, let $S: X \to Y$ be a linear bounded operator, and let $P: Y \supset D(P) \to Z$ be a linear closed operator with dense domain. If the domain $D(PS)$ of the product $PS: X \supset D(PS) \to Z$ is dense in X, then the adjoint operator $(PS)^*$ is equal to the weak closure of the product S^*P^* of their adjoint operators S^* and P^*.*

In Hilbert spaces, the weak closure of linear sets coincides with their strong closure. We apply Lemma 4.1 in the Hilbert spaces $X = Y = Z = \mathcal{H}_\tau$ to the bounded operator $Su = tu$ with domain $D(S) = \mathcal{H}_\tau$ and to the closed operator $Pg = dg/dt$ with the domain

$$D(P) = \{g \in \mathcal{H}_\tau: \; dg/dt \in \mathcal{H}_\tau, \; g(\tau) = 0\}.$$

Obviously, the adjoint operator is $S^* = S = t: \; \mathcal{H}_\tau \to \mathcal{H}_\tau$. Obviously, the adjoint operator of P is the operator $P^* = -d/dt: \; \mathcal{H}_\tau \to \mathcal{H}_\tau$ with domain

$$D(P^*) = \{\widetilde{v} \in \mathcal{H}_\tau: \; d\widetilde{v}/dt \in \mathcal{H}_\tau, \; \widetilde{v}(T) = 0\}.$$

Then, by Lemma 4.1, the adjoint operator of their product is equal to

$$(PS)^* = \overline{S^*P^*} = -\overline{t(d/dt)}: \; \mathcal{H}_\tau \supset D((PS)^*) \to \mathcal{H}_\tau.$$

Here and throughout the following, the bar above operator products stands for their strong closures.

After the use of Lemma 4.1, identity (4.4) acquires the form

$$-\int_\tau^T \left(\frac{1}{t}\frac{dh}{dt}, \overline{t\frac{d}{dt}}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right]\right) dt = \int_\tau^T e^{c(T-t)}\Phi(h,w)dt \qquad \forall h \in M_\tau.$$

We extend the resulting identity by passing to the limit from functions $h \in M_\tau$ to all functions $h \in \mathcal{H}_\tau$ such that $t^{-1}(dh/dt) \in \mathcal{H}_\tau$ and $h(t) = 0$, $t \in [0, \tau]$, set $h = w$, and obtain

$$-\int_\tau^T \left(\frac{1}{t}\frac{dw}{dt}, \overline{t\frac{d}{dt}}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right]\right) dt = \int_\tau^T e^{c(T-t)}\Phi(w,w)\,dt. \tag{4.6}$$

Note that, in the inner product on the left-hand side of this relation, the traditional integration by parts with respect to t from the right to the left for the evaluation of its double real part is impossible, since the function dw/dt is not differentiable with respect to t in $\mathcal{H}_\tau$. Therefore, we use Lemma 4.1 again but in the Hilbert spaces $X = Y = \mathcal{H}_\tau$ and $Z = \mathcal{H}_\tau \times H$ and with different operators S and P. The operator $S = A^{-1}(t)(T-t)e^{c(T-t)}$ with domain $D(S) = \mathcal{H}_\tau$ is linear and bounded in $\mathcal{H}_\tau$. Its adjoint operator $S^* = S : \mathcal{H}_\tau \to \mathcal{H}_\tau$ is bounded as well. The operator $P = \{\overline{t(d\cdot/dt), \cdot|_{t=\tau}}\} : \mathcal{H}_\tau \supset D(P) \to \mathcal{H}_\tau \times H$ with dense domain $D(\overline{t(d\cdot/dt)})$ is linear and closed. Obviously, the adjoint operator of P coincides with the adjoint operator of the operator $P_0 = \{t(d\cdot/dt), \cdot|_{t=\tau}\} : \mathcal{H}_\tau \supset D(P_0) \to \mathcal{H}_\tau \times H$ with dense domain $D(P_0) = \{g \in \mathcal{H}_\tau : dg/dt \in \mathcal{H}_\tau, \ g(T) = 0\}$. Before using Lemma 4.1, we find the adjoint operator of P.

Lemma 4.2. *Let an operator* $P_0 : \mathcal{H}_\tau \to \mathcal{H}_\tau \times H$ *act in accordance with the relation* $P_0 g = \{t(dg/dt), g(\tau)\} \in \mathcal{H}_\tau \times H$ *on functions in its domain*

$$D(P_0) = \{g \in \mathcal{H}_\tau : \ dg/dt \in \mathcal{H}_\tau, \ g(T) = 0\}.$$

Then its adjoint operator $P_0^* : \mathcal{H}_\tau \times H \to \mathcal{H}_\tau$ *has the domain*

$$D^* = \{\{k(t), \tau k(\tau)\} \in \mathcal{H}_\tau \times H : \ d(tk(t))/dt \in \mathcal{H}_\tau\},$$

and its values are given by the formula $P_0^*(\{k(t), \tau k(\tau)\}) = -d(tk(t))/dt$.

Proof. By the definition of the adjoint operator, a vector function $\{k(t), \varphi\} \in \mathcal{H}_\tau \times H$ belongs to the domain $D(P_0^*)$ of the adjoint operator P_0^* if there exists a function $w \in \mathcal{H}_\tau$ such that

$$\int_\tau^T \left(t\,\frac{dg(t)}{dt}, k(t)\right) dt + (g(\tau), \varphi) = \int_\tau^T (g(t), w(t))\, dt \qquad \forall g \in D(P_0). \tag{4.7}$$

In addition, by construction, we assume that $w(t) = P_0^*(\{k(t), \varphi\})$.

Let us prove the inclusion $D^* \subset D(P_0^*)$. If a vector function $\{k(t), \tau k(\tau)\}$ belongs to the set D^*, then by integrating once by parts with respect to t,

$$\int_\tau^T \left(t\,\frac{dg(t)}{dt}, k(t)\right) dt + (g(\tau), \tau k(\tau)) = \int_\tau^T \left(g(t), -\frac{d}{dt}(tk(t))\right) dt \qquad \forall g \in D(P_0),$$

we prove the existence of the function $w(t) = -d(tk(t))/dt \in \mathcal{H}_\tau$. Consequently, the vector function $\{k(t), \tau k(\tau)\}$ belongs to the set $D(P_0^*)$.

Let us prove the opposite embedding $D(P_0^*) \subset D^*$. Let identity (4.7) hold for some vector function $\{k(t), \varphi\} \in \mathcal{H}_\tau \times H$. If the function g belongs to the set $D_0(P_0) = \{g \in D(P_0) : \ g(\tau) = 0\}$, then from this identity, we obtain

$$\int_\tau^T \left(\frac{dg(t)}{dt}, tk(t)\right) dt = \int_\tau^T (g(t), w(t))\, dt \qquad \forall g \in D_0(P_0).$$

This, together with the well-known assertion on the adjoint operator, implies that $d(tk(t))/dt \in \mathcal{H}_\tau$ and $w(t) = -d(tk(t))/dt$. Therefore, by setting $w(t) = -d(tk(t))/dt$ in the identity (4.7) and by performing integration by parts once with respect to t, we obtain the identity

$$\int_\tau^T \left(t\,\frac{dg(t)}{dt}, k(t)\right) dt + (g(\tau), \varphi) = (g(\tau), \tau k(\tau)) + \int_\tau^T \left(t\,\frac{dg(t)}{dt}, k(t)\right) dt \qquad \forall g \in D(P_0).$$

Hence we obtain the relation $(g(\tau), (\varphi - \tau k(\tau))) = 0$ for all $g \in D(P_0)$, which, in the case of $g(t) = (T-t)(\varphi - tk(t))$, implies that $(T-\tau)|\varphi - \tau k(\tau)|^2 = 0$ and hence $\varphi = \tau k(\tau)$. The proof of Lemma 4.2 is complete.

Thus, by Lemma 4.2, the adjoint operator P^* is given by the formula

$$P^*(\{k(t), \tau k(\tau)\}) = -d(tk(t))/dt,$$

and its domain is

$$D(P)^* = \{\{k(t), \tau k(\tau)\} \in \mathcal{H}_\tau \times H: \ d(tk(t))/dt \in \mathcal{H}_\tau\}.$$

To use Lemma 4.1 on the left-hand side of relation (4.6), one should show that $t^{-1}(dw/dt) \in D((PS)^*)$, where $D((PS)^*)$ is the domain of the adjoint operator $(PS)^*$ of the product PS. To this end, it suffices to note that, on the right-hand side of the obvious identity

$$\begin{aligned}\int_\tau^T \left(t\frac{d}{dt}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dh}{dt}\right], \frac{1}{t}\frac{dw}{dt}\right) dt &= \int_\tau^T \left(A^{-1}(t)(T-t)e^{c(T-t)}\frac{d^2h}{dt^2}, \frac{1}{t}\frac{dw}{dt}\right) dt \\ &+ \int_\tau^T \left(\frac{dA^{-1}(t)}{dt}(T-t)e^{c(T-t)}\frac{dh}{dt}, \frac{1}{t}\frac{dw}{dt}\right) dt - \int_\tau^T \left(A^{-1}(t)e^{c(T-t)}\frac{dh}{dt}, \frac{1}{t}\frac{dw}{dt}\right) dt \\ &- c\int_\tau^T \left(A^{-1}(t)(T-t)e^{c(T-t)}\frac{dh}{dt}, \frac{1}{t}\frac{dw}{dt}\right) dt - \int_\tau^T \left(A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dh}{dt}, \frac{1}{t}\frac{dw}{dt}\right) dt \quad \forall h \in M_\tau,\end{aligned}$$

the first integral is estimated from above by the right-hand side of inequality (4.5), and the remaining integrals are also estimated by the right-hand side of inequality (4.5) with some other constant $c_4 > 0$, i.e., by the norm $\left(\int_\tau^T |t^{-1}(dh/dt)|^2 dt\right)^{1/2}$ for all $h \in M_\tau$.

Now we rewrite the left-hand side of relation (4.6) for almost all $\tau \in]0, T[$ in the form

$$\begin{aligned}-\int_\tau^T \left(\frac{1}{t}\frac{dw}{dt}, \overline{t\frac{d}{dt}}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right]\right) dt - \left(t\frac{1}{t}\frac{dw}{dt}, A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right)\Bigg|_{t=\tau} \\ + \left(\frac{dw}{dt}, A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right)\Bigg|_{t=\tau},\end{aligned} \tag{4.8}$$

and here, in the first two terms (without the minus), we use Lemma 4.1 for the operators $S = A^{-1}(t)(T-t)e^{c(T-t)}: \ \mathcal{H}_\tau \to \mathcal{H}_\tau$ and $P = \{\overline{t(d\cdot/dt), \cdot|_{t=\tau}}\}: \ \mathcal{H}_\tau \supset D(P) \to \mathcal{H}_\tau \times H$. As a result, we obtain the value of the adjoint operator

$$(PS)^*\left(\frac{1}{t}\frac{dw}{dt}\right) = \overline{S^*P^*}\left(\frac{1}{t}\frac{dw}{dt}\right) = -\overline{A^{-1}(t)(T-t)e^{c(T-t)}\left(\frac{d}{dt}t\right)}\left(\frac{1}{t}\frac{dw}{dt}\right). \tag{4.9}$$

Since

$$\begin{aligned}A^{-1}(t)(T-t)e^{c(T-t)}\frac{d}{dt}\left[t\frac{1}{t}\frac{dw}{dt}\right] &= t\frac{d}{dt}\left[\left\{A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\right\}\left\{t\frac{1}{t}\frac{dw}{dt}\right\}\right] \\ &- t\left[\frac{d}{dt}\left\{A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\right\}\right]\left\{t\frac{1}{t}\frac{dw}{dt}\right\}\end{aligned}$$

for sufficiently smooth functions w, by taking the closure of this difference of operator products, from (4.9), we find that the value of the adjoint operator $(P\cdot S)^*(t^{-1}(dw/dt))$ is equal to

$$\begin{aligned}&\overline{\left\{-t\frac{d}{dt}[A^{-1}(t)(T-t)e^{c(T-t)}] + t\left(\frac{d}{dt}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\right]\right)t\right\}}\left(\frac{1}{t}\frac{dw}{dt}\right) \\ &\quad = \left\{\overline{-t\frac{d}{dt}[A^{-1}(t)(T-t)e^{c(T-t)}]} + t\left(\frac{d}{dt}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\right]\right)t\right\}\left(\frac{1}{t}\frac{dw}{dt}\right),\end{aligned} \tag{4.10}$$

by virtue of the inclusion $t(d[A^{-1}(t)(T-t)e^{c(T-t)}(1/t)]/dt)(dw/dt) \in \mathcal{H}_\tau$. For the closure of the product of operators on the right-hand side of relation (4.10), we use the following lemma.

Lemma 4.3 [2]. *Let X and Y be Banach spaces. If $S_1: X \to X$ is a linear bounded operator, and $P_1: X \supset D(P) \to Y$ is a linear operator with domain $D(P_1)$ that admits closure $\overline{P_1}$, then their product $P_1S_1: X \to Y$ admits closure $\overline{P_1S_1}$, and $\overline{P_1S_1} \subset \overline{P_1}S_1$.*

An application of this lemma to the linear bounded operator

$$S_1 = A^{-1}(t)(T-t)e^{c(T-t)}: \mathcal{H}_\tau \to \mathcal{H}_\tau$$

and the linear unbounded operator $P_1 = t(d/dt): \mathcal{H}_\tau \to \mathcal{H}_\tau$, which has domain $D(P_1) = \{h \in \mathcal{H}_\tau: dh/dt \in \mathcal{H}_\tau,\ h(T) = 0\}$ and admits closure $\overline{P_1} = \overline{t(d/dt)}$, implies the relation

$$-\overline{t\frac{d}{dt}[A^{-1}(t)(T-t)e^{c(T-t)}]}\frac{1}{t}\frac{dw}{dt} = -\overline{t\frac{d}{dt}}[A^{-1}(t)(T-t)e^{c(T-t)}]\frac{1}{t}\frac{dw}{dt}. \tag{4.11}$$

As a result, from (4.9)–(4.11), we obtain

$$\begin{aligned}(PS)^*\left(\frac{1}{t}\frac{dw}{dt}\right) &= -\overline{t\frac{d}{dt}}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right] + \frac{dA^{-1}(t)}{dt}(T-t)e^{c(T-t)}\frac{dw}{dt}\\ &- A^{-1}(t)e^{c(T-t)}\frac{dw}{dt} - cA^{-1}(t)(T-t)e^{c(T-t)}\frac{dw}{dt} - A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}.\end{aligned} \tag{4.12}$$

By using formulas (4.12), from the representation (4.8), we obtain

$$\begin{aligned}&-2\,\mathrm{Re}\int_\tau^T\left(\frac{1}{t}\frac{dw}{dt}, -\overline{t\frac{d}{dt}}\left[A^{-1}(t)(T-t)e^{c(T-t)}\frac{1}{t}\frac{dw}{dt}\right]\right)dt = \frac{(T-\tau)}{\tau}e^{c(T-\tau)}\left|A^{-1/2}(\tau)\frac{dw(\tau)}{dt}\right|^2\\ &-\int_\tau^T e^{c(T-t)}(T-t)\left(\frac{dA^{-1}(t)}{dt}\frac{dw}{dt}, \frac{1}{t}\frac{dw}{dt}\right)dt + \int_\tau^T e^{c(T-t)}\left(A^{-1}(t)\frac{dw}{dt}, \frac{1}{t}\frac{dw}{dt}\right)dt\\ &+c\int_\tau^T e^{c(T-t)}(T-t)\left(A^{-1}(t)\frac{dw}{dt}, \frac{1}{t}\frac{dw}{dt}\right)dt + \int_\tau^T e^{c(T-t)}(T-t)\left(A^{-1}(t)\frac{1}{t}\frac{dw}{dt}, \frac{1}{t}\frac{dw}{dt}\right)dt\end{aligned} \tag{4.13}$$

for almost all $\tau \in]0, T[$.

By integrating by parts once with respect to t, we obtain

$$\begin{aligned}&2\,\mathrm{Re}\int_\tau^T e^{c(T-t)}\left(w, (T-t)\frac{1}{t}\frac{dw}{dt}\right)dt\\ &\quad= c\int_\tau^T e^{c(T-t)}(T-t)\frac{1}{t}|w|^2dt + \int_\tau^T e^{c(T-t)}\frac{1}{t}|w|^2\,dt + \int_\tau^T e^{c(T-t)}(T-t)\frac{1}{t^2}|w|^2dt.\end{aligned} \tag{4.14}$$

If, on both sides in relation (4.6), we take the doubled real part, then, by virtue of (4.13) and (4.14), we obtain

$$\frac{(T-\tau)}{\tau}e^{c(T-\tau)}\left|A^{-1/2}(\tau)\frac{dw(\tau)}{dt}\right|^2 + \int_\tau^T e^{c(T-t)}\left|A^{-1/2}(t)\frac{1}{\sqrt{t}}\frac{dw}{dt}\right|^2dt$$

$$= \int_\tau^T e^{c(T-t)}(T-t)\Psi(w,w)\,dt - \int_\tau^T e^{c(T-t)}\left|\frac{1}{\sqrt{t}}w\right|^2 dt$$
$$- \int_\tau^T e^{c(T-t)}(T-t)\left(\left|A^{-1/2}(t)\frac{1}{t}\,\frac{dw}{dt}\right|^2 + \left|\frac{1}{t}w\right|^2\right)dt, \tag{4.15}$$

where the sesquilinear form is given by

$$\Psi(w,w) = -3\left(\frac{dA^{-1}(t)}{dt}\,\frac{1}{\sqrt{t}}\,\frac{dw}{dt}, \frac{1}{\sqrt{t}}\,\frac{dw}{dt}\right) - 2\,\mathrm{Re}\left(\frac{d^2A^{-1}(t)}{dt^2}\,\frac{1}{\sqrt{t}}\,w, \frac{1}{\sqrt{t}}\,\frac{dw}{dt}\right)$$
$$- 2\,\mathrm{Re}\left(\frac{B(t)A^{-1}(t)}{t}\,\frac{1}{\sqrt{t}}\,\frac{dw}{dt}, A(t)A^{-1}(t)\,\frac{1}{\sqrt{t}}\,\frac{dw}{dt}\right) - 2\,\mathrm{Re}\left(\frac{B(t)}{t}\,\frac{dA^{-1}(t)}{dt}\,\frac{1}{\sqrt{t}}\,w, \frac{1}{\sqrt{t}}\,\frac{dw}{dt}\right)$$
$$- c\left|A^{-1/2}(t)\frac{1}{\sqrt{t}}\,\frac{dw}{dt}\right|^2 - c\left|\frac{1}{\sqrt{t}}w\right|^2.$$

By virtue of relations (1.4), (1.5), (1.7), and (1.8) and the inequality $2ab \le a^2 + b^2$, the form $\Psi(w,w)$ does not exceed the quantity

$$(3a_1 + a_2 + 2b_1 + b_2 - c)|A^{-1/2}(t)t^{-1/2}(dw/dt)|^2 + (a_2 + b_2 - c)|t^{-1/2}w|^2,$$

which is nonpositive for $c \ge c_5 = 3a_1 + a_2 + 2b_1 + b_2$ and can be omitted.

By relation (4.15) taken for $c = c_5$ and by estimating the left-hand side from below and the right-hand side from above by zero, we obtain the inequality $\int_\tau^T e^{c_5(T-t)}|A^{-1/2}(t)t^{-1/2}(dw/dt)|^2\,dt \le 0$, which implies that $dw/dt = 0$ for almost all $t \in\,]\tau, T[$ and for all $\tau \in [0, T[$; consequently, $v = 0$ in $\mathcal{H}$.

The uniqueness of strong solutions of the Cauchy problem (1.1), (1.2) and the estimate (4.1) for them follow from inequality (3.1). The proof of Theorem 4.1 is complete.

Remark 4.1. By using the well-known method of continuation with respect to a parameter, one can prove the assertion of Theorem 4.1 for the singular hyperbolic equation with the lower term

$$\frac{d^2u(t)}{dt^2} + \frac{B(t)}{t}\,\frac{du(t)}{dt} + A(t)u(t) + B_1(t)\frac{du(t)}{dt} + A_1(t)u(t) = f(t), \qquad t \in\,]0, T[$$

(possibly, with a higher value of c_1) provided that $B_1(t)$, $A_1(t)A^{-1/2}(t) \in L_\infty(]0,T[, \mathcal{L}(H))$.

Remark 4.2. The analysis of the proof shows that, under the assumptions of Theorem 4.1, the Cauchy problem (1.1), (1.2) is well posed in the spaces E with norm

$$\|u\|_E = \sup_{0<t<T}(|du(t)/dt|^2 + |A^{1/2}(t)u(t)|^2)^{1/2}$$

and $F = \mathcal{H}$, and inequality (4.1) holds for $c_1 = \exp(\max\{a_1, 2b_0 + 1\}T)/2$.

5. APPLICATIONS

By using Theorem 4.1 and Remark 4.1, one can prove the well-posed solvability of new mixed problems for singular hyperbolic partial differential equations with nonstationary boundary conditions. In the domain $G =]0, l[\times]0, T[$, consider the singular hyperbolic equation

$$u_{tt} + t^{-1}(b_1(x)u_{xt} + b_0(x)u_t) - (a(x)u_x)_x + b_2(x,t)u_t + a_0(x,t)u_x + a_1(x,t)u = f(x,t), \quad \{x,t\} \in G, \tag{5.1}$$

with the boundary conditions

$$u|_{x=0} = 0, \qquad [u_x + \beta(t)u]|_{x=l} = 0, \qquad t \in [0, T], \tag{5.2}$$

and the initial conditions

$$u|_{t=0} = 0, \qquad u_t|_{t=0} = 0, \qquad x \in\,]0, l[. \tag{5.3}$$

Theorem 5.1. *Let the coefficients of Eq.* (5.1) *satisfy the conditions*

$$\begin{aligned}
&a(x) \geq a_0 > 0, \quad x \in [0,l], \quad b_1(0) = 0, \quad b_1(l) \geq 0, \quad a(x) \in C^{(1)}[0,l], \quad b_i(x) \in C^{(i)}[0,l],\\
&b_2(x,t), a_i(x,t) \in C(\overline{G}), \quad i = 0,1, \quad \beta(t) \in C^{(2)}[0,T], \quad 0 \leq \beta(t) \leq c_6 t^{\alpha}, \quad \alpha \geq 1,\\
&|\beta'(t)| \leq c_7 t^{\gamma}, \quad \gamma \geq 1, \quad c_6, c_7 > 0, \quad t \in [0,T], \quad 2b_0(x) - b_1'(x) \geq 0,\\
&2\beta(t)b_0(l) + b_0'(l) \geq 0, \quad a'(x)b_0'(x) + a(x)b_0''(x) \leq 0,\\
&a(x)b_1'(x) - b_1(x)a'(x) + 2a(x)b_0(x) \geq 0, \quad \{x,t\} \in \overline{G},
\end{aligned} \tag{5.4}$$

where one (*respectively, two*) *primes stands for the first* (*respectively, second*) *derivatives of functions. Then the mixed problem* (5.1)–(5.3) *for all* $f \in \mathcal{F}(G)$ *has a unique strong solution* $u \in \mathcal{E}(G)$ *continuously depending on* f *in the spaces* $\mathcal{F}(G)$ *and* $\mathcal{E}(G)$ *with the corresponding norms* (2.1) *and* (2.2).

Proof. Let us verify conditions (A_0)–(A_2), (B_0), and (B_1) and the requirements of Remark 4.1 for problem (5.1)–(5.3) in the Hilbert space $H = L_2(0,l)$. Obviously, the linear operators $A(t)u = -(a(x)u_x)_x$ in the domains $D(A(t)) = \{u \in W_2^2(0,l) : \ u|_{x=0} = 0, \ [u_x + \beta(t)u]|_{x=l} = 0\}$, $t \in [0,T]$, are self-adjoint in $L_2(0,l)$ and, by virtue of the condition $a(x) \geq a_0 > 0$, satisfy inequality (1.3). On the entire space $L_2(0,l)$, they have bounded inverse operators

$$A^{-1}(t)g = -\int_0^x a(s)^{-1} \int_0^s g(\tau)\,d\tau\,ds + \left(q_1(t)\int_0^l g(s)\,ds + q_2(t)\int_0^l a(s)^{-1}\int_0^s g(\tau)\,d\tau\,ds\right)\overline{a}(x),$$

where $\overline{a}(y) = \int_0^y a(s)^{-1}\,ds$, $q_1(t) = (1 + a(l)\overline{a}(l)\beta(t))^{-1}$, $q_2(t) = a(l)\beta(t)(1 + a(l)\overline{a}(l)\beta(t))^{-1}$, and $a(s)^{-1} = (a(s))^{-1}$. Their boundedness is provided by the inequality $\|A^{-1}(t)h\|_0 \leq c_8\|h\|_0$ for all $h \in L_2(0,l)$, where $\|\cdot\|_0$ is the norm in $L_2(0,l)$ and $c_8 = l\overline{a}(l)\max_{t\in[0,T]}(1 + |q_1(t)| + \overline{a}(l)|q_2(t)|)$.

If the function $\beta(t)$ belongs to $C^{(1)}[0,T]$, then there exists a bounded derivative

$$\frac{dA^{-1}(t)}{dt}h = -q_3(t)\overline{a}(x)\int_0^l \overline{a}(s)h(s)\,ds$$

satisfying the inequality $\|(dA^{-1}(t)/dt)h\|_0 \leq c_9\|h\|_0$ for all $h \in L_2(0,l)$, where

$$q_3(t) = a(l)\beta'(t)(1 + a(l)\overline{a}(l)\beta(t))^{-2}, \quad c_9 = la(l)(\overline{a}(l))^{-2}\max_{t\in[0,T]}\left[|\beta'(t)|(1 + a(l)\overline{a}(l)\beta(t))^{-2}\right].$$

Let us prove the estimate (1.4) for it. By integrating by parts once with respect to t, we find that its left-hand side for $g = A(t)u$ is equal to

$$\begin{aligned}
&\int_0^l q_3(t)\left(\int_0^l (a(s)u'(s))'\overline{a}(s)\,ds\right)\overline{a}(x)(a(x)u'(x))'\,dx\\
&\qquad = q_3(t)\left(\int_0^l (-(a(x)u'(x))')\overline{a}(x)\,dx\right)^2 = q_3(t)\left(\int_0^l u'(x)\,dx - a(l)\overline{a}(l)u'(l)\right)^2.
\end{aligned}$$

By using the Cauchy–Schwarz and Minkowski inequalities, one can derive the inequality

$$\begin{aligned}
&\left|\int_0^l u'(x)\,dx - a(l)\int_0^l a(s)^{-1}\,ds\,u'(l)\right| \leq \int_0^l |u'(x)|\,dx + a(l)\overline{a}(l)|u'(l)|\\
&\qquad \leq c_{10}^{1/2}\left(\int_0^l a(x)|u'(x)|^2\,dx + \frac{a(l)}{1+\beta(t)}|u'(l)|^2\right)^{1/2} \leq c_{10}^{1/2}\|A^{1/2}(t)u\|_0,
\end{aligned} \tag{5.5}$$

where

$$\|A^{1/2}(t)u\|_0^2 = \frac{a(l)}{1+\beta(t)}|u'(l)|^2 + \frac{a(l)\beta(t)}{1+\beta(t)}|u(l)|^2 + \int\limits_0^l a(x)|u'(x)|^2\,dx \tag{5.6}$$

and $c_{10} = a(l)(\overline{a}(l))^2 \max_{t\in[0,T]}(1+\beta(t)) + \overline{a}(l)$. Hence we obtain the estimate

$$-((dA^{-1}(t)/dt)A(t)u, A(t)u)_0 \le a_1\|A^{1/2}(t)u\|_0^2$$

for all $u \in D(A(t))$, which is equivalent to the estimate (1.4) with constant

$$a_1 = a(l)\max_{t\in[0,T]}\{|\beta'(t)|(1+a(l)\overline{a}(l)\beta(t))^{-2}\}c_{10}.$$

By the symbol $(\cdot\,,\cdot)_0$, we denote the inner product in $L_2(0,l)$.

If the function $\beta(t)$ belongs to $C^{(2)}[0,T]$, then the second derivative

$$\frac{d^2A^{-1}(t)}{dt^2}\,h = -q_3'(t)\overline{a}(x)\int\limits_0^l \overline{a}(s)h(s)\,ds$$

satisfies the estimate (1.5) for the constant

$$a_2 = l^{1/2}a(l)\overline{a}(l)\max_{t\in[0,T]}\{|\beta''(t)(1+a(l)\overline{a}(l)\beta(t)) - 2a(l)\overline{a}(l)(\beta'(t))^2|(1+a(l)\overline{a}(l)\beta(t))^{-3}\}c_{10}^{1/2},$$

since the left-hand side of this estimate for $v = A(t)u$ and $u \in D(A(t))$, is equal to

$$\left|\int\limits_0^l q_3'(t)\int\limits_0^l g(s)\overline{a}(s)\,ds\,\overline{a}(x)(a(x)u'(x))'\,dx\right| = |q_3'(t)|\left|\int\limits_0^l g(s)\overline{a}(s)\,ds\right|\left|\int\limits_0^l (a(x)u'(x))'\overline{a}(x)\,dx\right|$$

$$\le l^{1/2}\overline{a}(l)|q_3'(t)|\left(\int\limits_0^l |g(s)|^2\,ds\right)^{1/2}\left|\int\limits_0^l u'(x)\,dx - a(l)\overline{a}(l)u'(l)\right|.$$

Here we estimate the right-hand side with the use of inequality (5.5) and obtain the equivalent estimate $|((d^2A^{-1}(t)/dt^2)g, A(t)u)_0| \le a_2\|g\|_0\|A^{1/2}(t)u\|_0$ for all $g \in L_2(0,l)$ and for all $u \in D(A(t))$. In particular, from the estimate (1.5), we find that the operators $d^2A^{-1}(t)/dt^2$ are bounded in $L_2(0,l)$ uniformly with respect to t.

If $b_i(x) \in C^{(i)}[0,l]$, $i = 0,1$, then, obviously, the operator $B(t) = b_1(x)(\partial/\partial x) + b_0(x)$ with domain $D(B(t)) = W_2^1(0,l)$ is subordinate to the operators $A^{1/2}(t)$ in $L_2(0,l)$ by virtue of (5.6); and if $b_1(l) \ge 0$ and $2b_0(x) - b_1'(x) \ge 0$, then the estimate (1.6) holds for the constant $b_0 = 0$, since, after integration by parts once with respect to x, its left-hand side is equal to

$$-\operatorname{Re}\int\limits_0^l (b_1(x)u'\overline{u} + b_0(x)|u|^2)\,dx = -\frac{b_1(l)}{2}|u(l)|^2 + \frac{1}{2}\int\limits_0^l (b_1'(x) - 2b_0(x))|u|^2\,dx \le 0.$$

Obviously, the set $\widetilde{D}(L(t))$ of problem (5.1)–(5.3) is dense in $\mathcal{H} = L_2(G)$.

If $b_1(0) = 0$, $b_0(x) \in C^{(2)}[0,l]$, $\beta(t) \le c_6t^\alpha$, $\alpha \ge 1$, $t \in [0,T]$, and inequalities (5.4) hold, then inequality (1.7) is true with constant $b_1 = (1/2)b_1(l)\max_{t\in[0,T]}[(1+\beta(t))t^{\alpha-1}]c_6$ since, after integrating by parts once with respect to x, its left-hand side is equal to

$$\operatorname{Re}\int\limits_0^l [b_1(x)u' + b_0(x)u]\,\overline{(a(x)u')'}\,dx = \frac{a(l)}{2}\left[b_1(l)\beta^2(t) - (2\beta(t)b_0(l) + b_0'(l))\right]|u(l)|^2$$

$$-\frac{1}{2}\int\limits_0^l (a(x)b_1'(x) - b_1(x)a'(x) + 2a(x)b_0(x))|u'|^2\,dx + \frac{1}{2}\int\limits_0^l (a'(x)b_0'(x) + a(x)b_0''(x))|u|^2\,dx.$$

If $|\beta'(t)| \le c_7 t^\gamma$, $\gamma \ge 1$, $t \in [0,T]$, then the operator

$$B(t)\frac{dA^{-1}(t)}{dt}g = -q_3(t)\int\limits_0^l g(s)\overline{a}(s)\,ds\,\frac{b_1(x)}{a(x)} - q_3(t)\overline{a}(x)\int\limits_0^l g(s)\overline{a}(s)\,ds\,b_0(x)$$

satisfies inequality (1.8) with constant

$$b_2 = \left(\int\limits_0^l \overline{a}^2(s)\,ds\right)^{1/2}\Bigg\{a(l)[b_1(l)+a(l)\overline{a}(l)|b_0(l)|]^2 \max_{t\in[0,T]}(1+\beta(t))$$
$$+\,a^2(l)\left[\left(\int\limits_0^l \frac{(b_1'(x))^2}{a(x)}\,dx\right)^{1/2} + \left(\int\limits_0^l \frac{b_1^2(x)(a'(x))^2}{a^3(x)}\,dx\right)^{1/2} + \left(\int\limits_0^l a(x)(b_0'(x))^2\overline{a}^2(x)\,dx\right)^{1/2}\right.$$
$$\left.+\left(\int\limits_0^l \frac{(b_0(x))^2}{a(x)}\,dx\right)^{1/2}\right]^2\Bigg\}^{1/2} \max_{t\in[0,T]}\left[(1+a(l)\overline{a}(l)\beta(t))^{-2}\,t^{\gamma-1}\right]c_7.$$

Indeed, the left-hand side of this inequality for $h = A(t)u$, $u \in D(A(t))$, is equal to the quantity

$$|q_3(t)|\left|\int\limits_0^l g(s)\overline{a}(s)\,ds\right|\left|\int\limits_0^l \left[\frac{b_1(x)}{a(x)} + b_0(x)\int\limits_0^x \frac{ds}{a(s)}\right]\overline{(a(x)u')'}\,dx\right|.$$

In the integral of the last absolute value of this quantity, we perform the single integration by parts with respect to x, use the Cauchy–Schwarz and Minkowski inequalities and obvious estimates, and find that it does not exceed the quantity

$$\Bigg\{\frac{1+\beta(t)}{a(l)}[b_1(l)+a(l)\overline{a}(l)|b_0(l)|]^2 + \left[\left(\int\limits_0^l \frac{(b_1'(x))^2}{a(x)}\,dx\right)^{1/2} + \left(\int\limits_0^l \frac{b_1^2(x)(a'(x))^2}{a^3(x)}\,dx\right)^{1/2}\right.$$
$$+\left(\int\limits_0^l a(x)(b_0'(x))^2\overline{a}^2(x)\,dx\right)^{1/2}$$
$$\left.+\left(\int\limits_0^l \frac{(b_0(x))^2}{a(x)}\,dx\right)^{1/2}\right]^2\Bigg\}^{1/2}\left(\frac{a(l)}{1+\beta(t)}|u'(l)|^2 + \int\limits_0^l a(x)|u'|^2\,dx\right)^{1/2}.$$

This implies the inequality

$$|(B(t)(dA^{-1}(t)/dt)g, A(t)u)_0| \le b_2 t\|g\|_0\|A^{1/2}(t)u\|_0$$

for all $g \in L_2(0,l)$ and for all $u \in D(A(t))$, which is equivalent to inequality (1.8). In particular, this inequality shows that the operators $B(t)(dA^{-1}(t)/dt)$ are bounded in $L_2(0,l)$ uniformly with respect to t.

Obviously, the operators $B_1(t)u = b_2(x,t)u$ and $A_1(t)u = a_1(x,t)(\partial u/\partial x) + a_2(x,t)u$ with domains $D(B_1(t)) = L_2(0,l)$ and $D(A_1(t)) = W_2^1(0,l)$, respectively, satisfy the requirements of Remark 4.1. The proof of Theorem 5.1 is complete.

Remark 5.1. All assumptions of Theorem 5.1 are satisfied by the functions $a(x) = x + l$, $b_1(x) = -x^2 + lx$, $b_0(x) = -(x^2/2) + lx + 2l$ for all $l > 0$, and $\beta(t) = c_{11}\sin^2 t$ for all $c_{11} > 0$. In Eq. (5.1), the coefficient $a(x)$ depends only on x for the simplification of the validity of assumptions of Theorem 4.1 but can depend on t as well.

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