Representation Varieties of the Fundamental Groups of Compact Non-Orientable Surfaces

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Abstract

We give a description of the varieties of n-dimensional representations and characters of fundamental groups of compact non-orientable surfaces.

1 Introduction

Let $\Gamma =< g_1, \dots, g_m >$ be a finitely generated group and $G \subset GL_n(K)$ a connected linear algebraic group defined over a field K which throughout the paper will be assumed to be algebraically closed and of characteristic zero. For any homomorphism $\rho : \Gamma \to G(K)$ the set of elements

$$(\rho(g_1), \dots, \rho(g_m)) \in G(K)^m = G(K) \times \cdots \times G(K)$$

satisfies evidently all the relations of Γ and thus the correspondence

$$\rho \to (\rho(g_1), \dots, \rho(g_m))$$

gives a bijection between points of the set $\mathrm{Hom}(\Gamma, G(K))$ and K-points of some affine K-variety $R(\Gamma, G) \subset G^m$ whose geometric structure does not depend on the choice of generators $g_1, \dots, g_m$ of Γ.

The variety $R(\Gamma, G)$ is usually called the *representation variety* of Γ into the algebraic group G. In the case $G = \mathrm{GL}_n(K)$ we will denote it simply by $R_n(\Gamma)$ and call it the variety of n-dimensional representations of Γ.

The group G acts on $R(\Gamma, G)$ by simultaneous conjugation and its orbits are in one-to-one correspondence with the equivalence classes of representations of Γ. Under this action orbits of G are not necessarily closed and so the set of orbits (also called the geometric quotient) is not an algebraic variety. However if G is a reductive group, then one can consider a categorical quotient $R(\Gamma, G)/G$ (see [11]), which is usually denoted by $X(\Gamma, G)$ and is called the *variety of characters* (for more details see [10]). By construction, its points parametrize closed G-orbits. For $G = \mathrm{GL}_n(K)$ an orbit of a representation ρ is closed iff ρ is fully reducible. It follows that the points of the variety $X_n(\Gamma) = X(\Gamma, \mathrm{GL}_n(K))$ are in one-to-one correspondence with the equivalence classes of fully reducible n-dimensional representations of Γ (see [10]).

If Γ is an arbitrary finitely generated group we know practically nothing about the structure of the varieties $R_n(\Gamma)$, $X_n(\Gamma)$. It has been studied for classes of infinite nilpotent and solvable groups (see [10], [15]) and in detail for the class of finite groups only. Recall that if $|\Gamma| < \infty$ then the description of $R_n(\Gamma)$, $X_n(\Gamma)$ is given by the classical representation theory of finite groups. Namely, every representation is fully reducible and up to equivalence there is only a finite number of irreducible representations and all of them are uniquely determined by their characters; in particular, $\dim X(\Gamma, G) = 0$, i.e. $X(\Gamma, G)$ is a finite set for all finite groups Γ.

For topological applications it is important to know the description of the varieties of n-dimensional representations and characters for those groups Γ which arise as fundamental groups of some natural classes of manifolds. At present the answer is known for fundamental groups Γ_g of compact orientable surfaces of genus g only. First Goldman [4] found the number of connected components of $R(\Gamma_g, \mathrm{SL}_2(\mathbb{C}))$ in real and complex topology