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MATHEMATICS

First-Order Differential Equations with Variable Domains of Piecewise Smooth Operator Coefficients

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This paper describes new results concerning the well-posedness (in the sense of Hadamard) of a first-order operator-differential equation with piecewise smooth operators defined in variable domains. The operators are not necessarily self-adjoint (in contrast to [1]) and do not necessarily have global smoothing operators with constant domains (in contrast to [2]) or local smoothing operators with variable domains (in contrast to [3]), but they have only majorizing operators with variable domains. For the first time, we prove the smoothness of weak solutions to such equations (Theorem 2). The weak well-posedness of two new mixed problems for variable-order partial differential equations is substantiated.

1. In a Hilbert space H with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, we consider the Cauchy problem

$$\frac{du(t)}{dt} + A(t)u = f, \quad t \in]0, T[; \quad u(0) = u_0, \quad (1)$$

where u and f are H-valued functions of t and $A(t)$ denotes linear unbounded operators in H with t-dependent domains $D(A(t))$ that satisfy the following conditions.

(I) For any $t \in [0, T]$, the operators $A(t)$ are closed in H and satisfy

$$[u]^2_{(t)} \equiv \operatorname{Re}(A(t)u + c_0 u, u) \geq c_1 |u|^2 \ \forall u \in D(A(t)), \quad (2)$$

$$\langle v \rangle^2_{(t)} \equiv \operatorname{Re}(A^*(t)v + c_0 v, v) \geq c_1 |v|^2 \quad \forall v \in D(A^*(t)), \quad c_0 \geq 0, \quad c_1 > 0, \quad (3)$$

where $A^*(t)$ with domains $D(A^*(t))$ are the adjoints of $A(t)$ in H.

(II) On each partial interval $I_r = [t_r, t_{r+1}[$ of the partition $[0, T[= \coprod_{r=0}^{R} I_r$, the inverses $A_0^{-1}(t) \in \mathcal{B}(I_r, \mathcal{L}(H))$ of the operators $A_0(t) = A(t) + c_0 I$ are strongly continuous with respect to $t \in I_r$ in H and, for almost all (a.a.) $t \in I_r$ in H, there exists a weak derivative $dA_0^{-1}(t)/dt \in L_\infty(I_r, \mathcal{L}(H))$ and

$$\left|dA_0^{-1}(t)/dtg, h\right| \leq c_2(r)[A_0^{-1}(t)g]_{(t)}|h| \quad \forall g, h \in H, \quad c_2(r) \geq 0, \quad r = 0, 1, \ldots, R. \quad (4)$$

(III) On each interval I_r ($r = 0, 1, \ldots, R-1$), the operators $A^*(t)$ are subordinated to the adjoints $B^*(t)$ (with domains $D(B^*(t))$ in H) of some closed operators $B(t)$ with t-dependent domains $D(B(t))$.

(a) Conditions I and II hold for the operators $B^*(t)$ instead of $A_0(t)$ on each interval I_r ($r = 0, 1, \ldots, R-1$) separately.

(b) The inverses $B^{-1}(t) \in \mathcal{B}(I_r, \mathcal{L}(H))$ of the operators $B(t)$ are weakly continuous with respect to $t \in I_r$ in H and, for a.a. $t \in I_r$, they have a weak derivative $\frac{dB^{-1}(t)}{dt} \in L_\infty(I_r, \mathcal{L}(H))$ in H such that

$$-\operatorname{Re}\left(\varphi, \left(\frac{dB^{-1}(t)}{dt}\right)B(t)\varphi\right) \leq c_3(r)|\varphi|[\operatorname{Re}(B(t)\varphi, \varphi)]^{1/2}, \quad c_3 \geq 0, \quad r = 0, 1, \ldots, R-1 \quad (5)$$

for a.a. $t \in I_r$ $\forall \varphi \in D(B(t))$.

(c) At all the discontinuity points t_r of $A_0^{-1}(t)$,

$$D(A^*(t_r)) \subset D(B^*(t_r - 0)), \quad r = 1, 2, \ldots, R, \quad (6)$$

where $D(B^*(t_r - 0))$ are the domains of the left extensions $B^*(t_r - 0)$ of $B^*(t)$ ($t < t_r$) to the points $t_r \in I_r$.

Condition III(a) implies that the operators $B^{*-1}(t)$ have maximal accretive extensions $B^*(t_r - 0)$ by conti-

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nuity: for all $g \in H$, we set $v(t_r) \in D(B^*(t_r - 0))$ and $g = B^*(t_r - 0)v(t_r)$ if $v(t) = B^{*-1}(t)g \to v(t_r)$ strongly in H. Obviously, $B^*(t)v(t) = g \to g$ strongly in H as $t \xrightarrow{<} t_r$, $r = 1, 2, \dots, R$.

Let H_t^{*-} be the antiduals of the Hilbert spaces H_t^{*+} obtained as the closure of $D(A^*(t))$ with respect to the Hermitian norms $\langle\cdot\rangle_{(t)}$ in (3).

Definition 1. A function $u \in \mathcal{H} = L_2(]0, T[, H)$ is called a weak solution to Cauchy problem (1) with a right-hand side $f \in \mathcal{H}^{*-} = L_2(]0, T[, H_t^{*-})$ and an initial function $u_0 \in H$ if u satisfies the identity

$$\int_0^T \left\{ (u(t), A^*(t)\varphi(t)) - \left(u(t), \frac{d\varphi(t)}{dt}\right) \right\} dt = \int_0^T \langle f(t), \varphi(t) \rangle_{(t)} dt + (u_0, \varphi(0)) \quad \forall \varphi \in \Phi. \tag{7}$$

Here, $\Phi = \{\varphi \in H\text{: } \varphi(t) \in D(A^*(t)),\ t \in [0, T]$; weak derivative $\frac{d\varphi}{dt}$, $A^*(t)\varphi \in \mathcal{H}$; $\varphi(T) = 0\}$, where $\langle \cdot, \cdot \rangle_{(t)}$ are the antiduality forms between H_t^{*+} and H_t^{*-}.

We analyze the weak well-posedness of Cauchy problem (1) with piecewise smooth unbounded operators and examine the local smoothness of its weak solutions. Additionally, we analyze the well-posedness of previously unstudied mixed problems for partial differential equations with time-discontinuous coefficients in the equations and with piecewise smooth coefficients in the boundary conditions. The existence and uniqueness of weak solutions are stated in Theorem 1; their stability is treated in Corollary 1; their local smoothness, in Corollary 2; and the well-posedness of new mixed problems is stated in Theorems 3 and 4.

Remark 1. If the operators $A(t)$ satisfy Conditions I and II on I_r, then $B(t) = A_0(t)$ obviously satisfies Condition III(b) on I_r.

2. First, we prove the existence and uniqueness of weak solutions.

Theorem 1. *If Conditions* I–III *are satisfied, then, for any* $f \in \mathcal{H}^{*-}$ *and* $u_0 \in H$, *Cauchy problem* (1) *has a unique weak solution* $u \in \mathcal{H}$.

In [4], Lions' projection theorem [1, p. 37] was used to show that, under Condition I, for any $f \in \mathcal{H}^{*-}$ and $u_0 \in H$, Cauchy problem (1) has a weak solution $u \in \mathcal{H}$. According to Definition 1, its uniqueness will be proved if we show that the identity

$$\int_0^T \left\{ (u(t), A^*(t)\varphi(t)) - \left(u(t), \frac{d\varphi(t)}{dt}\right) \right\} dt = 0 \quad \forall \varphi \in \Phi, \tag{8}$$

which follows from (7) at $f = 0$ and $u_0 = 0$, implies that the weak solution $u(t)$ vanishes for a.a. t on each interval I_r ($r = 0, 1, \dots, R$) separately.

To show that $u = 0$ almost everywhere on $I_0 = [0, t_1[$, we set in (8) $\varphi(t) = 0$ for $t \in [t_1, T[$ and $\varphi(t) = A_0^{*-1}(t)w(t)$ for $t \in [0, t_1[$, where $w(t) = -\int_t^{t_1} e^{-2cs} u(s)ds$ for $t \in [0, t_1]$ or $u(t) = e^{2ct}\left(\frac{dw(t)}{dt}\right)$ for $t \in [0, t_1[$ and $w(t_1) = 0$. Next, taking the real part gives

$$\operatorname{Re}\int_0^{t_1} e^{2ct}\left\{\left(\frac{dw}{dt}, w\right) - \left(\frac{dw}{dt}, c_0 A_0^{*-1}(t)w + \frac{dA_0^{*-1}(t)}{dt}w + A_0^{*-1}(t)\frac{dw}{dt}\right)\right\}dt = 0, \quad c \ge 0. \tag{9}$$

Here, the existence of a weak derivative $\frac{dA_0^{*-1}(t)}{dt} = \left(\frac{dA_0^{-1}(t)}{dt}\right)^* \in L_\infty(I_r, \mathcal{L}(H))$ for the inverses $A_0^{*-1}(t)$ of the operators $A_0^*(t) = A^*(t) + c_0 I$ follows from Condition II. The first scalar product in (9) is integrated once by part with respect to t and inequalities (2) and (4) and the obvious relation $[A_0^{-1}(t)g]_{(t)} = \langle A_0^{*-1}(t)g \rangle_{(t)}$ are applied to the second scalar product to obtain

$$c\int_0^{t_1} e^{2ct}|w|^2 dt + \int_0^{t_1} e^{2ct}\left\langle A_0^{*-1}(t)\frac{dw}{dt}\right\rangle_{(t)}^2 dt - (c_0 c_1^{-1/2} + c_2(0))\int_0^{t_1} e^{2ct}\left\langle A_0^{*-1}(t)\frac{dw}{dt}\right\rangle_{(t)} |w| dt \le 0.$$

For $c > \frac{(c_0 c_1^{-1/2} + c_2(0))^2}{4}$, combining this relation with the Cauchy–Schwarz inequality, we conclude that $w(t) = 0$ and $u(t) = 0$ for a.a. $t \in I_0$.

To show that $u = 0$ almost everywhere on $I_1 = [t_1, t_2[$, we substitute into (8)

$$\varphi(t) = \begin{cases} 0, & t_2 \le t \le T \\ A_0^{*-1}(t)w(t), & t_1 \le t < t_2 \\ \psi_0(t), & 0 \le t < t_1, \end{cases} \tag{10}$$

where $w(t) = -\int_t^{t_2} e^{-2cs}u(s)ds$ for $t \in [t_1, t_2[$ or $u(t) = e^{2ct}\left(\frac{dw(t)}{dt}\right)$ for $t \in [t_1, t_2[$ and $w(t_2) = 0$, while $\psi_0(t)$ is a weak solution to the reverse-time Cauchy problem

$$\frac{d\psi_0(t)}{dt} - B^*(t)\psi_0(t) = 0, \quad 0 < t < t_1;$$

$$\psi_0(t_1) = v_{0,0} = A_0^{*-1}(t_1)w(t_1).$$

By the substitution $\tau = t_1 - t$, this problem is reduced to the direct-time Cauchy problem

$$\frac{d\theta_0(\tau)}{d\tau} + B_0^*(\tau)\theta_0(\tau) = 0, \ B_0^*(\tau) = B^*(t_1 - \tau),$$
$$0 < \tau < t_1; \quad \theta_0(0) = v_{0,0} \tag{11}$$

for the new function $\theta_0(\tau) = \psi_0(t)$.

For the first time, we prove the following smoothness theorem.

Theorem 2. *Let Conditions* III(a) *and* III(b) *be satisfied.*

Then, for any $f_r \in L_2(I_r, W^*(t))$ *and* $v_{r,0} \in W^*(t_r)$, *the Cauchy problems*

$$\frac{dv_r(\tau)}{d\tau} + B_0^*(\tau)v_r(\tau) = f_r(\tau),$$

$$\tau \in \,]t_r, t_{r+1}[; \quad v_r(t_r) = v_{r,0}, \quad r = 0, 1, \ldots, R-1$$

have unique weak solutions $v_r \in \mathcal{H}_r = L_2(I_r, H)$ *with the properties*

$$v_r(\tau) \in D(B_0^*(\tau)), \quad \tau \in I_r;$$

$$\frac{d^k v_r(\tau)}{d\tau^k} \in L_2(I_r, W^{*1-k}(\tau)), \quad k = 0, 1,$$

$$k = 0, 1, \quad r = 0, 1, \ldots, R-1.$$

Here, the Hilbert space $W^{*1}(T)$ *is the domain* $D(B_0^*(\tau))$ *of the operators* $B_0^*(\tau) = B^*(t_{r+1} + t_r - \tau)$ *with Hermitian norms* $\|v\|_{(\tau)} = |B_0^*(\tau)v|$, $\tau \in I_r$.

Embedding (6) with $r = 1$ in Condition III(c) implies that the initial function $v_{0,0}$ in Cauchy problem (11) belongs to $DB_0^*(0))$. To this Cauchy problem, we apply Theorem 2 with $f_r = 0$, $v_{r,0} = v_{0,0}$, and $r = 0$. As a result, we conclude that its weak solution satisfies $\theta_0(\tau) \in D(B_0^*(\tau))$ for $\tau \in I_0$ and $\frac{d\theta_0(\tau)}{d\tau}$, $B_0^*(\tau)\theta_0(\tau) \in L_2(I_0, H)$. Therefore, $\varphi(t)$ given by (10) belongs to the set Φ. Substituting this $\varphi(t)$ into (8) and taking the real part yields

$$\text{Re}\int_{t_1}^{t_2} e^{2ct}\left\{\left(\frac{dw}{dt}, w\right) - \left(\frac{dw}{dt}, c_0A_0^{*-1}(t)w + \frac{dA_0^{*-1}}{dt}w + A_0^{*-1}(t)\frac{dw}{dt}\right)\right\}dt = 0, \quad c \geq 0.$$

Proceeding as before, we find that $u(t) = 0$ for a.a. $t \in I_1$, etc., with the only difference being that the functions $\varphi(t) = \eta_r(t)h_r(t)$ are substituted into (8) on the intervals $I_{r,r} = 2, 3, \ldots, R$. Here,

$$h_r(t) = \begin{cases} 0, & t_{r+1} \leq t \leq T \\ A_0^{*-1}(t)w(t), & t_r \leq t < t_{r+1} \\ \psi_{r-1}(t), & t_{r-1} \leq t < t_r \\ 1, & 0 \leq t < t_{r-1}, \end{cases}$$

$$w(t) = -\int_t^{t_{r+1}} e^{-2cs}u(s)ds, \quad t \in I_r;$$

$\psi_{r-1}(t)$ are weak solutions to the reverse-time Cauchy problem

$$\frac{d\psi_{r-1}(t)}{dt} - B^*(t)\psi_{r-1}(t) = 0, \quad t_{r-1} < t < t_r;$$

$$\psi_{r-1}(t_r) = v_{r-1,0} = A_0^{*-1}(t_r)w(t_r);$$

and the cutoff functions $\eta_r(t) \in C^\infty(\mathbb{R})$ are defined as

$$\eta_r(t) = \begin{cases} 0 \ \text{for} \ t \in [0, t_{r-1}] \\ 1 \ \text{for} \ t \in [t_r, T] \end{cases}$$

$r = 2, 3, \ldots, R$. As a result, after $R + 1$ steps, if $R < +\infty$, we conclude that $u(t) = 0$ for a.a. $t \in I_r$, $r = 0, 1, \ldots, R$. The case $R = +\infty$ is reduced to $R < +\infty$ by using the concept of a complete measure in $]0, T[$, since weak solutions to Cauchy problem (1) exist on almost the entire interval $]0, T[$ without Conditions II and III.

Corollary 1. *If Conditions* I–III *are satisfied, then, for any* $f \in \mathcal{H}^{*-}$ *and* $u_0 \in H$, *the weak solutions* $u \in \mathcal{H}$ *to Cauchy problem* (1) *satisfy*

$$\int_0^T |u(t)|^2 dt \leq \frac{4}{c_1}\left(\int_0^T \langle f(t)\rangle_{(-t)}^2 dt + |u_0|^2\right),$$

$\langle\cdot\rangle_{(-t)}$ *is the norm in* H_t^{*-}.

Corollary 2. *Suppose that Conditions* I *and* II *are satisfied and inequality* (5) *holds for* $B(t) = A_0^*(t)$. *Then, by Theorem* 2, *for any* $f \in L_2(I_r, W(t))$ *and* $u(t_r) \in W(t_r)$, *where* $W(t)$ *are the sets* $D(A(t))$ *equipped with the* *graph norm of the operators $A_0(t)$, the weak solutions to Cauchy problem* (1) *on I_r are smooth*; *i.e.*,

$$u(t) \in D(A(t)), \quad t \in I_r;$$
$$\frac{du(t)}{dt}, \quad A(t)u(t) \in L_2(I_r, H), \quad r = 0, 1, \dots, R.$$

Using the results of [5], we analyzed the well-posedness of the following mixed problems.

3. In a bounded domain $G =]0, T[\times]0, l[$, consider the equation

$$\frac{\partial u(t,x)}{\partial t} + A(t)u(t,x) = f(t,x), \quad \{t, x\} \in G, \quad (12)$$

where $A(t) = a(t)\frac{\partial^3}{\partial x^3}$ for $t \in [0, t_1[$ and $A(t) = -a(t)\frac{\partial^6}{\partial x^6}$ for $t \in [t_1, T]$, with the t-dependent boundary conditions

$$\frac{\partial^2 u(t,0)}{\partial x^2} = -a_1(t)u(t,0),$$
$$\frac{\partial^2 u(t,l)}{\partial x^2} = a_2(t)u(t,l), \quad \frac{\partial u(t,l)}{\partial x} = 0, \quad (13)$$

$$\frac{\partial^5 u(t,0)}{\partial x^5} = -a_1(t)\frac{\partial^3 u(t,0)}{\partial x^3},$$
$$\frac{\partial^2 u(t,0)}{\partial x^2} = a_1(t)u(t,0), \quad \frac{\partial u(t,0)}{\partial x} = 0,$$
$$\frac{\partial^5 u(t,l)}{\partial x^5} = a_2(t)\frac{\partial^3 u(t,l)}{\partial x^3}, \quad (14)$$
$$\frac{\partial^2 u(t,l)}{\partial x^2} = -a_2(t)u(t,l), \quad \frac{\partial^4 u(t,l)}{\partial x^4} = 0$$

on the intervals $[0, t_1[$ and $[t_1, T]$, respectively, and with the initial conditions

$$u(0,x) = u_0(x), \quad 0 < x < l. \quad (15)$$

Theorem 3. *Suppose that the coefficients in* (12)–(14) *satisfy* $0 < a_0 \le a(t)$, $t \in [0, T]$, $0 < a_{i,0} \le a_i(t)$, $t \in [0, t_1[$, $a_i(t) \ge 0$, $t \in [t_1, T]$, $a(t) \in C^{(1)}(I_r)$, $a_i(t) \in C[0, T] \cap C^{(1)}(I_r)$, $I = 1, 2$, $r = 0, 1, \dots$.

Then, for every $f \in L_2(]0, T[, W^{*-}_{2,t}(0, l))$ and $u_0 \in L_2(0, l)$, mixed problem (12)–(15) *has a unique weak solution $u \in L_2(G)$. Here, $W^{*-}_{2,t}(0, l)$ are the antiduals of the Hilbert spaces $W^{*+}_{2,t}(0, l)$, which are the closures of the corresponding sets $D(A^*(t))$ with respect to the Hermitian norms* $\langle v \rangle_{(t)} = (\mathrm{Re}(A_0^*(t)v, v)_0)^{1/2}$.

4. In the bounded domain G, consider the variable-order equation

$$\frac{\partial u(t,x)}{\partial t} + A_r(t)u(t,x) = f(t,x),$$
$$x \in]0, l[, \quad t \in I_r, \quad r = 1, 2, \dots, \quad (16)$$

where

$$A_r(t)u(x) = a_r(t)(-1)^{r+m_r}\frac{\partial^{2(r+m_r)}u(x)}{\partial x^{2(r+m_r)}},$$

$$t \in I_r = \left[\frac{T(r-1)}{r}, \frac{Tr}{r+1}\right[$$

on the infinite partition $[0, T[= \coprod_{r=1}^{\infty} I_r$ (m_r is a nondecreasing integer number sequence), with the boundary conditions

$$\frac{\partial^{2i+1}u(t,0)}{\partial x^{2i+1}} - a_1(t)\frac{\partial^{2i}u(t,0)}{\partial x^{2i}} = 0,$$
$$\frac{\partial^{2i+1}u(t,l)}{\partial x^{2i+1}} + a_2(t)\frac{\partial^{2i}u(t,l)}{\partial x^{2i}} = 0, \quad t \in I_r, \quad (17)$$
$$i = 0, 1, \dots, r + m_r - 1, \quad r = 1, 2, \dots.$$

Theorem 4. *Suppose that the coefficients satisfy* $a < a_{r,0} \le a_r(t)$, $t \in I_r$, $a_i(t) \ge 0$, $t \in [0, T]$, $a_r(t) \in C^{(1)}(I_r)$, $a_i(t) \in C[0, T] \cap C^{(1)}(I_r)$, $i = 1, 2$; $r = 1, 2, \dots$.

Then, for every $f \in L_2(I_r, W^{-r-m_r}_{2,t}(0, l))$, $r = 1, 2, \dots$, and $u_0 \in L_2(0, l)$, mixed problem (16), (17), (15) *has a unique weak solution $u \in L_2(G)$. Here, $W^{-r-m_r}_{2,t}(0, l)$ are the antiduals of the Hilbert spaces $W^{r+m_r}_{2,t}(0, l)$, which are the closures of the corresponding sets $D(A^r(t))$ with respect to the Hermitian norms* $\langle v \rangle_{(t)} = (A_r^*(t)v + v, v)_0^{1/2}$.

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