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Invariant f-Structures on Naturally Reductive Homogeneous Spaces

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Abstract—We study invariant metric f-structures on naturally reductive homogeneous spaces and establish their relation to generalized Hermitian geometry. We prove a series of criteria characterizing geometric and algebraic properties of important classes of metric f-structures: nearly Kähler, Hermitian, Kähler, and Killing structures. It is shown that canonical f-structures on homogeneous Φ-spaces of order k (homogeneous k-symmetric spaces) play remarkable part in this line of investigation. In particular, we present the final results concerning canonical f-structures on naturally reductive homogeneous Φ-spaces of order 4 and 5.

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1. INTRODUCTION

Affinor structures on smooth manifolds, i.e., smooth tensor fields of type $(1, 1)$ realized as fields of endomorphisms acting in the tangent bundle of a manifold, are classical objects of investigation in differential geometry. There are very many types of such structures (see, e.g., the survey paper [1]), and there arise many new structures. At the same time, traditionally, almost complex, almost product, almost contact structures, and some other structures are intensively studied. Since 1960s, of great concern are f-structures introduced by K. Yano [2] ($f^3 + f = 0$), which generalize almost complex and almost contact structures. The *metric f-structures* (manifolds with an f-structure and an agreeing (pseudo)Riemannian metric) include the classes of *almost Hermitian structures* and *metric almost contact structures*, which play an important role in differential geometry and many its applications. In turn, metric f-structures are important objects of study in *generalized Hermitian geometry*, an area of modern differential geometry developed since the middle of 1980s (see, e.g., [3–5]).

In the differential geometry of homogeneous manifolds of Lie groups, the study of invariant affinor structures is a fundamental line of research. The classical theories of Riemannian and Hermitian symmetric spaces (see, e.g., [6]) became the base in the search for new classes of homogeneous spaces with invariant structures. In this respect, *homogeneous Φ-spaces* (see, e.g., [7–9]), which are also called *generalized symmetric spaces* [10], play an important role. First of all, homogeneous Φ-spaces of order 3 (homogeneous 3-symmetric spaces [10]) having a canonical almost complex structure ([11], [8]) gave a wide range of invariant almost Hermitian structures which, in the case of a naturally reductive metric, are nearly Kähler [12–14]. Later, it was discovered that regular Φ-spaces (in particular, homogeneous k-symmetric spaces) have many canonical structures of classical type including almost complex structures and f-structures [15, 16]. This gave a possibility not only to enlarge the number of known homogeneous manifolds with invariant almost Hermitian structures but also to present, with the use of canonical f-structures, first classes of invariant examples in generalized Hermitian geometry [17–23]. Note that an important role here was played by homogeneous Φ-spaces of order 4 and 5 endowed with naturally reductive metrics, and, in addition, a significant analogy was established with classical results of N. A. Stepanov, J. A. Wolf, A. Gray, V. F. Kirichenko in Hermitian geometry.

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The class of naturally reductive homogeneous spaces is intensively studied in differential geometry and its applications. Being a wide generalization of Riemannian globally symmetric spaces, such spaces possess the following property: All geodesics on these spaces are homogeneous, i.e., can be obtained as trajectories of one-parameter subgroups of the isometry group [24]. Later it was established that there are other spaces with the same property, which gave rise to a new line of research devoted to finding geodesically orbital spaces (g. o. spaces). Not considering the history of this problem and the extensive bibliography, we mention only the recent papers [25, 26]. Note that the majority of known invariant Einstein metrics on compact homogeneous spaces are naturally reductive (see the survey paper [27]).

In this paper, we study invariant metric f-structures on naturally reductive homogeneous spaces. We establish a series of criteria characterizing geometric and algebraic properties of important classes of metric f-structures: nearly Kähler, Hermitian, Kähler, and Killing structures. We present the final results concerning canonical f-structures on naturally reductive homogeneous Φ-spaces of order 4 and 5.

Some of results of this paper were partially announced in [17, 19].

2. METRIC f-STRUCTURES ON MANIFOLDS

First we consider briefly some facts of generalized Hermitian geometry concerning metric f-structures on smooth manifolds. For detailed information and general approaches, we refer to [3–5].

Recall that an f-structure on a manifold M is a field of endomorphisms f acting in the tangent bundle of M and satisfying the condition $f^3 + f = 0$ [2]. The number $r = \dim \operatorname{Im} f$ is constant for all points of M [28], it is called the *rank* of the f-structure. In addition, the number $\dim \operatorname{Ker} f = \dim M - r$ is usually called the *defect* of an f-structure and denoted by $\operatorname{def} f$. One can easily see that the particular cases $\operatorname{def} f = 0$ and $\operatorname{def} f = 1$ of f-structures lead to almost complex and almost contact structures respectively.

Let M be an f-manifold and $\mathfrak{X}(M)$ the module of smooth vector fields on M. Then $\mathfrak{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = \operatorname{Im} f$ and $\mathcal{M} = \operatorname{Ker} f$ are complementary distributions, which are usually called the *first* and the *second fundamental distributions* of an f-structure, respectively. It is clear that the endomorphisms $l = -f^2$ and $m = \mathrm{id} + f^2$ are the complementary projectors onto the distributions $\mathcal{L}$ and $\mathcal{M}$, respectively. Note that the restriction F of an f-structure to $\mathcal{L}$ is an almost complex structure, i.e., $F^2 = -\mathrm{id}$.

The Nijenhuis tensor of an f-structure is defined by [5]

$$N(X, Y) = \frac{1}{4}([fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]), \tag{1}$$

where $X, Y \in \mathfrak{X}(M)$. The condition $N = 0$ is a criterion of integrability of an f-structure ([29], P. 20).

Consider now some notions used in *generalized Hermitian geometry*. Such a geometry appeared (see, e.g., [3] and [4]) as a natural consequence of the development of the Hermitian geometry and the theory of almost contact metric structures together with their numerous applications. The basic object of study in this geometry is a *generalized almost Hermitian structure* (briefly, *GAH-structure*) of arbitrary *rank* r on a (pseudo)Riemannian manifold (M, g) [3, 4]. We will not formulate here the detailed definition of this general notion and restrict ourselves to consideration of an important special case of GAH-structures of rank 1, metric f-structures, which contain the class of almost Hermitian structures.

Recall that an f-structure on a (pseudo)Riemannian manifold $(M, g = \langle \cdot, \cdot \rangle)$ is called a *metric f-structure* if $\langle fX, Y \rangle + \langle X, fY \rangle = 0$, $X, Y \in \mathfrak{X}(M)$ [4]. In this case, the triple (M, g, f) is called a *metric f-manifold*. It is clear that the tensor field $\Omega(X, Y) = \langle X, fY \rangle$ is skew-symmetric, i.e., Ω is a 2-form on M. Ω is called the *fundamental form* of a metric f-structure [3, 4]. One can easily see that the partial cases $\operatorname{def} f = 0$ and $\operatorname{def} f = 1$ of metric f-structures lead to *almost Hermitian structures* and *almost contact metric structures*, respectively.

Let M be a metric f-manifold. Then the first and the second fundamental distributions $\mathcal{L} = \operatorname{Im} f$ and $\mathcal{M} = \operatorname{Ker} f$ are mutually orthogonal. Note that, in the case when the restriction of the metric g to $\mathcal{L}$ is nondegenerate, the restriction (F, g) of the metric f-structure to $\mathcal{L}$ is an almost Hermitian structure, i.e., $F^2 = -\mathrm{id}$, $\langle FX, FY \rangle = \langle X, Y \rangle$, $X, Y \in \mathcal{L}$.

A fundamental part in the geometry of generalized almost Hermitian structures (in particular, in the geometry of metric f-structures) is played by a special tensor T of type $(2,1)$ called the *compositional tensor*. Using the tensor T, one can introduce in $\mathfrak{X}(M)$ the structure of so-called *associated Q-algebra* by the formula [3, 4]

$$X * Y = T(X,Y).$$

Based on natural properties of the associated Q-algebras [3, 4], this enables us to introduce into consideration certain classes of GAH-structures. Note that, for metric f-manifolds, the explicit expression of the tensor T was given in [4]:

$$T(X,Y) = \frac{1}{4} f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y), \tag{2}$$

where ∇ is the Levi-Civita connection of the (pseudo)Riemannian manifold (M,g), $X, Y \in \mathfrak{X}(M)$.

Below we list the main classes of metric f-structures and indicate the properties defining these classes:

$\mathbf{Kf}$	*Kähler f-structure*	$\nabla f = 0$;
$\mathbf{Hf}$	*Hermitian f-structure*	$T(X,Y) = 0$, i.e., $\mathfrak{X}(M)$ is an abelian Q-algebra;
$\mathbf{G_1f}$	f-structure of class G_1 or $G_1 f$-structure	$T(X,X) = 0$, i.e., $\mathfrak{X}(M)$ is an anticommutative Q-algebra;
$\mathbf{QKf}$	*quasi-Kähler f-structure*	$\nabla_X f + T_X f = 0$;
$\mathbf{Kill\,f}$	*Killing f-structure*	$\nabla_X(f)X = 0$;
$\mathbf{NKf}$	*nearly Kähler f-structure or NKf-structure*	$\nabla_{fX}(f)fX = 0$.

The classes $\mathbf{Kf}$, $\mathbf{Hf}$, $\mathbf{G_1f}$, and $\mathbf{QKf}$ (in a more general situation) were introduced in [4] (see also [30]). Killing f-manifolds $\mathbf{Kill\,f}$ were defined and studied in [31], [32]. The class $\mathbf{NKf}$ was defined in [19, 20].

There are the following obvious inclusions between the classes of metric f-structures:

$$\mathbf{Kf} = \mathbf{Hf} \cap \mathbf{QKf}; \quad \mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1f}; \quad \mathbf{Kf} \subset \mathbf{Kill\,f} \subset \mathbf{NKf} \subset \mathbf{G_1f}.$$

It is important to note that, in the case $f = J$, we obtain the corresponding classes of almost Hermitian structures [33]. For example, for $f = J$, classes $\mathbf{Kill\,f}$ and $\mathbf{NKf}$ coincide with the well-known class $\mathbf{NK}$ of *nearly Kähler structures*.

Note that a Kähler f-structure is always integrable, which coincides with the case of classical Kähler structure J. Indeed, since the connection ∇ is torsion-free, we have $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. Then the Nijenhuis tensor $N(X,Y)$ for the f-structure can be written in the form ([5], P. 410)

$$N(X,Y) = \frac{1}{4}\,(\nabla_{fX}(f)Y - f\nabla_X(f)Y - \nabla_{fY}(f)X + f\nabla_Y(f)X),$$

which implies that, for a Kähler f-structure, $N(X,Y) = 0$.

At the same time, generally speaking, a Hermitian f-structure is not integrable, which differs it essentially from a classical Hermitian structure. Recall in this connection that the fact that an almost Hermitian structure (g, J) is Hermitian is equivalent to the fact that it is integrable (see, e.g., [33]).

We also note that Killing f-structures are often defined by the requirement that the fundamental form Ω is a Killing form, i.e., $d\Omega = \nabla\Omega$ [31, 34]. It is easy to show that this requirement is equivalent to the above given condition ([5], P. 419).

For particular classes of metric f-structures, the compositional tensor T can be written in a simpler form. More exactly, the following statement holds.

Lemma 1. *The compositional tensor T of any NKf-structure on a smooth manifold $(M, \langle\cdot,\cdot\rangle, f)$ is of the form*

$$T(X,Y) = \frac{1}{2} f\nabla_{fX}(f)(fY), \tag{3}$$

where $X, Y \in \mathfrak{X}(M)$.

Proof. On polarization, the condition defining NKf-structures can be written in the form

$$\nabla_{fX}(f)(fY) + \nabla_{fY}(f)(fX) = 0.$$

In addition, one can easily check that any f-structure satisfies the identity (see, e.g., [34])

$$f\nabla_X(f)(f^2Y) + f^2\nabla_X(f)(fY) = 0.$$

Using the above indicated equalities, we obtain

$$-f\nabla_{f^2X}(f)(f^2Y) = f^2\nabla_{f^2X}(f)(fY) = -f^2\nabla_{fY}(f)(f^2X) = f^3\nabla_{fY}(f)(fX) = f\nabla_{fX}(f)(fY).$$

The latter equality allows us rewrite (2) in the form (3). □

Note that formula (3) generalizes a formula obtained earlier (with the use of the same method) in [34] for Killing f-manifolds (see also [5]).

3. NATURALLY REDUCTIVE SPACES WITH INVARIANT METRIC f-STRUCTURES

Now we pass to consideration of invariant metric f-structures on (pseudo)Riemannian homogeneous spaces.

Let G be a connected Lie group, H its closed subgroup, and $g = \langle\cdot,\cdot\rangle$ an invariant (pseudo)Riemannian metric on the homogeneous space G/H. As usual, we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups G and H, respectively. Assume that G/H is a reductive homogeneous space, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the reductive decomposition of the Lie algebra $\mathfrak{g}$. We identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at $o = H$. Then an invariant metric g is completely determined by its value at o. For convenience, we will use the same notation for an invariant metric $\langle\cdot,\cdot\rangle$ on G/H and its value at o. This agreement will also be used for all other invariant structures on G/H, in particular, for invariant f-structures.

Any invariant f-structure on G/H gives a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where the subspaces $\mathfrak{m}_1 = \operatorname{Im} f$ and $\mathfrak{m}_2 = \operatorname{Ker} f$ determine completely the first and the second fundamental distributions, respectively.

Let now $(G/H, g = \langle\cdot,\cdot\rangle, f)$ be a homogeneous reductive space with invariant (pseudo)Riemannian metric $\langle\cdot,\cdot\rangle$ and invariant f-structure. This means that, for all $X, Y \in \mathfrak{m}$,

$$\langle fX, Y\rangle + \langle X, fY\rangle = 0. \tag{4}$$

In addition, in this case, the subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal with respect to the metric $\langle\cdot,\cdot\rangle$.

Recall that $(G/H, \langle\cdot,\cdot\rangle)$ is called a *naturally reductive* space with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ [24] if

$$\langle [X,Y]_\mathfrak{m}, Z\rangle = \langle X, [Y,Z]_\mathfrak{m}\rangle \tag{5}$$

for all $X, Y, Z \in \mathfrak{m}$. Here the index $\mathfrak{m}$ denotes as usual the projection of vectors from $\mathfrak{g}$ to $\mathfrak{m}$ with respect to the indicated reductive decomposition.

Consider now some of the above listed classes of invariant metric f-structures on naturally reductive homogeneous spaces.

3.1. Invariant NKf-Structures

Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a homogeneous reductive space with invariant naturally reductive metric $\langle\cdot,\cdot\rangle$ and invariant metric f-structure.

As it is known [20], in the case of a naturally reductive space, the fulfillment of the condition $[fX, f^2X]_{\mathfrak{m}} = 0$ for all $X \in \mathfrak{m}$ is a criterion for an invariant metric f-structure to belong to the class **NKf**. Polarizing the above equality, we arrive at the criterion

$$[fX, f^2Y]_{\mathfrak{m}} = [f^2X, fY]_{\mathfrak{m}}, \tag{6}$$

where $X, Y \in \mathfrak{m}$. This equality is equivalent to the following one:

$$[f^2X, f^2Y]_{\mathfrak{m}} = -[fX, fY]_{\mathfrak{m}}. \tag{7}$$

Lemma 2. *For an invariant NKf-structure on a naturally reductive space $(G/H, \langle\cdot,\cdot\rangle, f)$, the following relation holds*:

$$f([X, fY]_{\mathfrak{m}}) = f^2([X, f^2Y]_{\mathfrak{m}}), \tag{8}$$

where $X, Y \in \mathfrak{m}$.

Proof. Using (4), (5), and (7), we obtain, for all $X, Y, Z \in \mathfrak{m}$, $\langle f([X, fY]_{\mathfrak{m}}), Z\rangle = -\langle [X, fY]_{\mathfrak{m}}, fZ\rangle = -\langle X, [fY, fZ]_{\mathfrak{m}}\rangle = \langle X, [f^2Y, f^2Z]_{\mathfrak{m}}\rangle = \langle [X, f^2Y]_{\mathfrak{m}}, f^2Z\rangle = \langle f^2([X, f^2Y]_{\mathfrak{m}}), Z\rangle$.

Since the metric $\langle\cdot,\cdot\rangle$ is nondegenerate on $\mathfrak{m}$, this implies (8). □

Let us compute the compositional tensor T for an NKf-structure in the case under consideration.

Theorem 1. *The compositional tensor T of an invariant NKf-structure on a naturally reductive space $(G/H, \langle\cdot,\cdot\rangle, f)$ is of the form*

$$2T(X, Y) = -f^2([fX, fY]_{\mathfrak{m}}) = f^2([f^2X, f^2Y]_{\mathfrak{m}}), \tag{9}$$

where $X, Y \in \mathfrak{m}$.

Proof. The expression of the tensor T for an NKf-structure on a smooth manifold is indicated in Lemma 1. Since $\nabla_X(f)Y = \nabla_X fY - f\nabla_X Y$ for smooth vector fields X and Y, in the case of a reductive homogeneous space, using the traditional technique of special vector fields in a neighborhood of the point $o = H \in G/H$, we obtain

$$\nabla_X(f)Y = \alpha(X, fY) - f\alpha(X, Y).$$

Here α is the Nomizu function of an invariant affine connection ∇ on G/H and $X, Y \in \mathfrak{m}$ [35]. Since the Levi-Civita connection, for naturally reductive spaces, is defined by the formula $\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$, we arrive at the equality

$$\nabla_X(f)Y = \frac{1}{2}([X, fY]_{\mathfrak{m}} - f([X, Y]_{\mathfrak{m}})), \quad X, Y \in \mathfrak{m}.$$

Using now Lemma 2, we obtain $\nabla_{fX}(f)fY = \frac{1}{2}([fX, f^2Y]_{\mathfrak{m}} - f([fX, fY]_{\mathfrak{m}})) = \frac{1}{2}([fX, f^2Y]_{\mathfrak{m}} - f^2([fX, f^2Y]_{\mathfrak{m}})) = \frac{1}{2}(1 - f^2)([fX, f^2Y]_{\mathfrak{m}})$. Taking into account the latter equality and applying Lemmas 1 and 2 and Eq. (7), we have $2T(X, Y) = f\nabla_{fX}(f)fY = \frac{1}{2}f(1 - f^2)([fX, f^2Y]_{\mathfrak{m}}) = f([fX, f^2Y]_{\mathfrak{m}}) = f^2([fX, f^3Y]_{\mathfrak{m}}) = -f^2([fX, fY]_{\mathfrak{m}}) = f^2([f^2X, f^2Y]_{\mathfrak{m}})$, which proves (9). □

As usual, we will denote by the indices 1 and 2 the projections of vectors from $\mathfrak{g}$ to $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

Theorem 2. *Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a naturally reductive homogeneous space with invariant NKf-structure. f is a Hermitian f-structure if and only if the following relation holds*:

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}. \tag{10}$$

Proof. Note first of all that formula (9) for the compositional tensor T can be written in the form

$$2T(X,Y) = -[X_1, Y_1]_1. \tag{11}$$

Indeed, for any $X, Y \in \mathfrak{m}$, we obtain $2T(X,Y) = f^2([f^2X, f^2Y]_{\mathfrak{m}}) = f^2([-X_1, -Y_1]_{\mathfrak{m}}) = -[X_1, Y_1]_1$. Hermitian f-structures are defined by the condition $T(X,Y) = 0$ for all $X, Y \in \mathfrak{m}$. By (11), this condition takes the form $[X_1, Y_1]_1 = 0$, which is equivalent to the inclusion $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$. □

As a particular case of the above statement, we obtain the following corollary.

Corollary 1. An invariant NK-structure on a naturally reductive homogeneous space $(G/H, \langle\cdot,\cdot\rangle, J)$ is a Kähler structure if and only if G/H is a locally symmetric space (i.e., $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$).

Proof. In fact, in the case $f = J$, condition (10) takes the form $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, i.e., G/H is a locally symmetric space. Therefore, a Hermitian structure J on G/H is a Kähler structure. □

Note that the statement of Corollary 1 was first proved (in somewhat other formulation) in [13].

Remark 1. One of the statements of Theorem 2 is valid in a stronger formulation. More precisely, as it has been proved in [22], condition (10) implies that an invariant metric f-structure on any reductive homogeneous space $(G/H, g)$ is Hermitian for an arbitrary invariant (pseudo)Riemannian metric g (not necessarily naturally reductive).

3.2. Invariant Kähler f-Structures

Consider invariant Kähler f-structures on naturally reductive homogeneous spaces. Several characteristic conditions take place.

Theorem 3. *Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a naturally reductive homogeneous space with invariant metric f-structure. The following conditions are equivalent*:

1) *f is a Kähler f-structure*;
2) $[X, fY]_{\mathfrak{m}} = f([X, Y]_{\mathfrak{m}})$ *for all* $X, Y \in \mathfrak{m}$;
3) $[\mathfrak{m}, \mathfrak{m}_1] \subset \mathfrak{h}$ *and* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$.

Proof. 1) $\Longleftrightarrow$ 2). The condition $\nabla f = 0$ for an invariant f-structure on a reductive homogeneous space, in terms of the Nomizu function α of the invariant affine connection ∇, takes the form $\alpha(X, fY) - f\alpha(X,Y) = 0$, where $X, Y \in \mathfrak{m}$. Since the space under consideration is naturally reductive, we have $\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}$. Therefore, $\frac{1}{2}[X, fY]_{\mathfrak{m}} - f(\frac{1}{2}[X,Y]_{\mathfrak{m}}) = 0$, which is equivalent to condition 2).

2) $\Longleftrightarrow$ 3). Let condition 2) hold. Note that condition 2) implies the equality

$$[X, fY]_{\mathfrak{m}} = [fX, Y]_{\mathfrak{m}}. \tag{12}$$

Indeed, letting $Y = X$ in 2), we have $[X, fX]_{\mathfrak{m}} = 0$ for all $X \in \mathfrak{m}$. Polarization of this equality gives (12). On the other hand, using (4), (5), and (12), for any $X, Y, Z \in \mathfrak{m}$, we obtain $\langle f([X,Y]_{\mathfrak{m}}), Z\rangle = -\langle [X,Y]_{\mathfrak{m}}, fZ\rangle = -\langle X, [Y, fZ]_{\mathfrak{m}}\rangle = -\langle X, [fY, Z]_{\mathfrak{m}}\rangle = -\langle [X, fY]_{\mathfrak{m}}, Z\rangle$. Hence, by virtue of the nondegeneracy of the metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, we arrive at the equality

$$f([X,Y]_{\mathfrak{m}}) = -[X, fY]_{\mathfrak{m}}. \tag{13}$$

Now, from condition 2) and Eq. (13) it follows that $f([X,Y]_{\mathfrak{m}}) = 0 = [X, fY]_{\mathfrak{m}}$. Since X and Y are arbitrary, this implies the inclusions: $[\mathfrak{m}, \mathfrak{m}_1] \subset \mathfrak{h}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$. Conversely, if relations 3) hold, then, obviously, equality 2) also holds. □

Remark 2. Condition 3) of Theorem 3 can be written in an equivalent form

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}, \;\; [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}, \;\; [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}.$$

Considering the special case $f = J$ of Theorem 3, we arrive at the following statement.

Corollary 2. An invariant almost Hermitian structure J on a naturally reductive homogeneous space $(G/H, \langle\cdot,\cdot\rangle, J)$ is a Kähler structure if and only if G/H is a locally symmetric space (i.e., $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$).

Note that this statement is a sharpening of Corollary 1.

3.3. Invariant Killing f-Structures

Now we consider invariant Killing f-structures on a naturally reductive space $(G/H, \langle\cdot,\cdot\rangle)$. As it is known [20], for such a space, the fulfillment of the condition $[X, fX]_{\mathfrak{m}} = 0$ for all $X \in \mathfrak{m}$ is a criterion for an invariant metric f-structure to be a Killing structure. Polarizing this equality, we write the above mentioned criterion in the form

$$[X, fY]_{\mathfrak{m}} = [fX, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}. \tag{14}$$

Lemma 3. *For an invariant Killing f-structure on a naturally reductive space $(G/H, \langle\cdot,\cdot\rangle, f)$, for all $X, Y \in \mathfrak{m}$, the following relations hold*:

$$f([X, Y]_{\mathfrak{m}}) = -[X, fY]_{\mathfrak{m}} = -[fX, Y]_{\mathfrak{m}}. \tag{15}$$

Proof. The arguments that establish the validity of the first equality, in fact, repeat a part of the proof of Theorem 3. Namely, using (4), (5), and (14), for any $X, Y, Z \in \mathfrak{m}$, we obtain

$$\langle f([X,Y]_{\mathfrak{m}}), Z\rangle = -\langle [X,Y]_{\mathfrak{m}}, fZ\rangle = -\langle X, [Y, fZ]_{\mathfrak{m}}\rangle = -\langle X, [fY, Z]_{\mathfrak{m}}\rangle = -\langle [X, fY]_{\mathfrak{m}}, Z\rangle.$$

Since the metric $\langle\cdot,\cdot\rangle$ is nondegenerate on $\mathfrak{m}$, taking into account criterion (14), we arrive at Eqs. (15). □

Theorem 4. *For an invariant Killing f-structure on a naturally reductive space $(G/H, \langle\cdot,\cdot\rangle, f)$, the Nijenhuis tensor N and the compositional tensor T are of the form*

$$N(X,Y) = [fX, fY]_{\mathfrak{m}} = -[f^2X, f^2Y]_{\mathfrak{m}} = f^2([X,Y]_{\mathfrak{m}}) = 2T(X,Y),$$

where $X, Y \in \mathfrak{m}$.

Proof. For an invariant metric f-structure on a reductive homogeneous space, the Nijenhuis tensor N defined by Eq. (1) is computed by the formula $N(X,Y) = \frac{1}{4}([fX, fY]_{\mathfrak{m}} - f([fX, Y]_{\mathfrak{m}}) - f([X, fY]_{\mathfrak{m}}) + f^2([X,Y]_{\mathfrak{m}}))$, where $X, Y \in \mathfrak{m}$. Taking into account Eqs. (14) and (15), for a Killing f-structure, we obtain

$$N(X,Y) = \frac{1}{4}([fX, fY]_{\mathfrak{m}} + [fX, fY]_{\mathfrak{m}} + [fX, fY]_{\mathfrak{m}} + [fX, fY]_{\mathfrak{m}}) = [fX, fY]_{\mathfrak{m}}.$$

Let us compute now the compositional tensor T. Since $\mathbf{Kill\,f} \subset \mathbf{NKf}$, we will use Lemma 1. Similarly to the reasoning from Theorem 1, using (15) and (14), in the case under consideration, we obtain $\nabla_X(f)Y = \frac{1}{2}([X, fY]_{\mathfrak{m}} - f([X,Y]_{\mathfrak{m}})) = -f([X,Y]_{\mathfrak{m}})$, $X, Y \in \mathfrak{m}$. Then $2T(X,Y) = f\nabla_{fX}(f)fY = f(-f([fX, fY]_{\mathfrak{m}})) = -f^2([fX, fY]_{\mathfrak{m}}) = -[fX, f^3Y]_{\mathfrak{m}} = [fX, fY]_{\mathfrak{m}}$. Another expression for the tensor T can be obtained, for example, with the use of Eq. (7): $2T(X,Y) = [fX, fY]_{\mathfrak{m}} = -[f^2X, f^2Y]_{\mathfrak{m}}$. Finally, in accordance with (15), we have one more representation $2T(X,Y) = [fX, fY]_{\mathfrak{m}} = -f([fX, Y]_{\mathfrak{m}}) = f^2([X,Y]_{\mathfrak{m}})$. □

Theorem 5. *Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a naturally reductive space with invariant Killing f-structure. Then the following relations hold*:

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}.$$

In particular, each of the fundamental distributions of a Killing f-structure defines an invariant totally geodesic foliation of the manifold G/H.

Proof. Let us prove the first relation. The subspace $\mathfrak{m}_1$ is characterized by the condition $f^2|_{\mathfrak{m}_1} = -\,\mathrm{id}$. Taking any $X, Y \in \mathfrak{m}$ and using Lemma 3, we obtain $f^2([fX, fY]_{\mathfrak{m}}) = [f^3X, fY]_{\mathfrak{m}} = -[fX, fY]_{\mathfrak{m}}$. Hence it follows that $[fX, fY]_{\mathfrak{m}} \in \mathfrak{m}_1$, i.e., $[fX, fY] \in \mathfrak{m}_1 \oplus \mathfrak{h}$. Since X and Y are arbitrary, we have $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}$.

Let us prove the second relation. Let $X \in \mathfrak{m}_2$, $Y \in \mathfrak{m}$. Then $f([X,Y]_{\mathfrak{m}}) = -[fX, Y]_{\mathfrak{m}} = -[0, Y]_{\mathfrak{m}} = 0$. This means that $[X,Y]_{\mathfrak{m}} \in \mathfrak{m}_2 = \mathrm{Ker}\, f$, i.e., $[\mathfrak{m}_2, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$, which implies, in particular, the second inclusion.

Finally, let us prove the third relation. For any $fX \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_2$, from the above obtained relation $[\mathfrak{m}_2, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$ it follows that $[fX, Y] \in \mathfrak{m}_2 \oplus \mathfrak{h}$. On the other hand, we have $f^2([fX, Y]_{\mathfrak{m}}) = [f^3X, Y]_{\mathfrak{m}} = -[fX, Y]_{\mathfrak{m}}$. Consequently, $[fX, Y]_{\mathfrak{m}} \in \mathfrak{m}_1$, i.e., $[fX, Y] \in \mathfrak{m}_1 \oplus \mathfrak{h}$. The two inclusions obtained imply $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$.

Let us discuss now the properties of the fundamental distributions of an f-structure defined by the subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$. As it is known, any f-structure generates on a manifold an almost product structure P as follows: $P = 2f^2 + \mathrm{id}$. In addition, the vertical and the horizontal distributions of the structure P are defined by the subspaces $\mathfrak{m}_2$ and $\mathfrak{m}_1$, respectively. What is more, one can easily show that, for a metric f-structure on a (pseudo)Riemannian manifold, the above constructed structure P is a (pseudo)Riemannian almost product structure, i.e., $\langle PX, PY\rangle = \langle X, Y\rangle$. Note that, the obtained inclusions, in the case of a naturally reductive space, are precisely the criterion [36] of the fact that each of the distributions generates an invariant totally geodesic foliation on G/H. ☐

Remark 3. It is known [31] that the second fundamental distribution of an f-structure on an arbitrary Killing f-manifold M is involutive and its leaves are totally geodesic submanifolds in M. In other words, the information on this distribution obtained in Theorem 5 is valid in the general situation. On the other hand, it is mentioned in [31] that the first fundamental distribution on a Killing f-manifold of the so-called basic type [31] is not involutive. Since, by Theorem 5, the distribution generated by the subspace $\mathfrak{m}_1$ is involutive, we arrive at the following conclusion.

Corollary 3. There are no nontrivial invariant Killing f-structures of basic type on a naturally reductive homogeneous space $(G/H, \langle\cdot,\cdot\rangle)$.

The above mentioned fact is a wide generalization of the corresponding result of A. S. Gritsans obtained for Riemannian globally symmetric spaces.

Remark 4. The expression of the compositional tensor T for invariant Killing f-structures in Theorem 4 can also be obtained by means of a detailed consideration of the result of Theorem 1 with the use of the first inclusion of Theorem 5. In fact, $2T(X, Y) = -f^2([fX, fY]_{\mathfrak{m}}) = [fX, fY]_{\mathfrak{m}}$.

The following theorem is one of the basic results on invariant Killing f-structures.

Theorem 6. *Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a naturally reductive space with invariant Killing f-structure. The following conditions are equivalent*:

1) *f is a Hermitian f-structure*;
2) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$;
3) $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$;
4) *f is integrable*;
5) *f is a Kähler f-structure.*

Proof. 1) $\Longleftrightarrow$ 2). From Theorem 4 it follows that $T(X, Y) = 0$ if and only if $[fX, fY]_{\mathfrak{m}} = 0$ for all $X, Y \in \mathfrak{m}$, which is equivalent to the inclusion $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$.

1) $\Longleftrightarrow$ 3). Using Theorem 4, consider the following representation of the tensor T: $2T(X, Y) = f^2([X, Y]_{\mathfrak{m}})$. Now the condition $T(X, Y) = 0$ is equivalent to the equality $f^2([X, Y]_{\mathfrak{m}}) = 0$ for all $X, Y \in \mathfrak{m}$, which is equivalent to the inclusion $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$.

1) $\Longleftrightarrow$ 4). This assertion is obvious since, by Theorem 4, the tensor T equals zero if and only if $N = 0$.

2) $\Longleftrightarrow$ 5). Assume that condition 2) holds. Since the f-structure under consideration is a Killing structure, by Theorem 5, we have the inclusions $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}$, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$. Thus, we arrive at the inclusions $[\mathfrak{m}, \mathfrak{m}_1] \subset \mathfrak{h}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$. Now, by item 3 of Theorem 3, it follows that the f-structure is a Kähler structure. The converse implication is obvious by virtue of item 3 of Theorem 3. ☐

Remark 5. As a particular case of this theorem (for $f = J$), we again arrive at the statement of Corollary 1.

In conclusion, we formulate one more result demonstrating the "degree of difference" of Killing f-structures from NKf-structures in the case of naturally reductive spaces.

Theorem 7 ([23]). *Let $(G/H, \langle\cdot,\cdot\rangle, f)$ be a naturally reductive space with invariant metric f-structure. The following conditions are equivalent*:

1) *f is a Killing f-structure*;

2) *f is an NKf-structure for which* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$.

4. CANONICAL f-STRUCTURES ON HOMOGENEOUS Φ-SPACES

The classes of invariant metric f-structures considered above can be efficiently realized on special families of homogeneous manifolds as well as in the form of concrete examples. In particular, homogeneous Φ-spaces endowed with canonical f-structures compose a wide class of invariant structures in generalized Hermitian geometry. We give a brief description of some results in the framework of the indicated area. First we need to list some facts concerning homogeneous Φ-spaces and canonical structures on such spaces. For more detailed information we refer to [7, 9, 15].

Let Φ be an automorphism of a connected Lie group G, G^Φ the subgroup of fixed points of Φ, and G^Φ_o the connected component of the unity e of the subgroup G^Φ. A homogeneous space G/H is called a *homogeneous Φ-space* if the closed Lie subgroup H in G satisfies the condition $G^\Phi_o \subset H \subset G^\Phi$. Let $A = \varphi - \mathrm{id}$, where $\varphi = d\Phi_e$ is the corresponding automorphism of the Lie algebra $\mathfrak{g}$. In this case, the Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ consists of φ-fixed vectors from $\mathfrak{g}$. A homogeneous Φ-space G/H is called a *regular Φ-space* if $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$ ([7, 9, 15]). A fundamental property of regular Φ-spaces is that these spaces are reductive [7], and the above indicated decomposition of the Lie algebra $\mathfrak{g}$ is a reductive decomposition. This decomposition is called the *canonical reductive decomposition* [7] of a reductive Φ-space G/H. Another fundamental result is as follows: all homogeneous Φ-spaces of *order* k ($\Phi^k = \mathrm{id}$) are regular [7]. These spaces are also called *homogeneous k-symmetric spaces* [10].

Note that the canonical reductive complement $\mathfrak{m} = A\mathfrak{g}$ is a φ-invariant subspace in $\mathfrak{g}$. Denote by θ the restriction of φ to $\mathfrak{m}$. An invariant affinor structure F on a regular Φ-space G/H is called *canonical* if its value at o is a polynomial of θ: $F = F(\theta)$ [15]. The canonical structures form a commutative subalgebra $\mathcal{A}(\theta)$ in the algebra $\mathcal{A}$ of all invariant affinor structures on a homogeneous space G/H. A very important feature of the algebra $\mathcal{A}(\theta)$ is that it contains a significant number of structures of classical type (almost product and almost complex structures, f-structures of classical and hyperbolic type), which are completely described in [15, 16]. In particular, for homogeneous Φ-spaces of order k, explicit computational formulas have been obtained. For example, all canonical f-structures can be given by formulas

$$f = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin \frac{2\pi m j}{k} \right) (\theta^m - \theta^{k-m}), \tag{16}$$

where $u = \begin{cases} n, & \text{if } k = 2n+1, \\ n-1, & \text{if } k = 2n, \end{cases}$, $\zeta_j \in \{-1, 0, 1\}$, and there are nonzero numbers among ζ_j [15]. In particular, for $\zeta_j \in \{-1, 1\}$, formula (16) gives an explicit expression for all canonical almost complex structures J on G/H (on condition that the spectrum of θ does not contain -1).

It is interesting that, for a homogeneous symmetric Φ-space ($\Phi^2 = \mathrm{id}$), the algebra $\mathcal{A}(\theta)$ is trivial, i.e., it consists of scalar structures, only. In the case $k = 3, 4, 5$, the general formulas for classical canonical structures are considered in detail in [15]. These structures include the classical canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous Φ-space of order 3, which was first discovered in [11] (see also [8, 12]). On a homogeneous Φ-space of order 4, there are (up to a sign) one canonical almost product structure $P = \theta^2$ and one canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ [15]. As for homogeneous Φ-spaces of order 5, such a space admits (in the case of maximal spectrum of θ), up to a sign, one canonical almost product structure P, two canonical almost complex structures J_1 and J_2, and two

f-structures f_1 and f_2 [15]. We do not give here explicit expressions for these structures, but mention that the fundamental distributions of the canonical f-structures are related as follows:

$$\mathfrak{m}_1 = \operatorname{Im} f_1 = \operatorname{Ker} f_2, \quad \mathfrak{m}_2 = \operatorname{Im} f_2 = \operatorname{Ker} f_1, \quad \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2.$$

Let a homogeneous Φ-space G/H of order k be endowed with a (pseudo)Riemannian metric generated by a symmetric bilinear form $g = \langle\cdot,\cdot\rangle$ on $\mathfrak{m} \times \mathfrak{m}$ which is invariant with respect to the subgroup $\mathrm{Ad}_G(H)$ and the operator θ. Such a metric is invariant not only with respect to the group G, but with respect to generalized symmetries of the homogeneous Φ-space G/H as well. It is known [23] that all canonical f-structures on $(G/H, g)$ agree with this metric, i.e., are invariant metric f-structures. In particular, canonical almost complex structures J are invariant almost Hermitian structures.

In the case of a semisimple Lie group G, the so-called standard metric induced by the Killing form of the Lie algebra $\mathfrak{g}$ is a classical example of a metric g with the above indicated properties. We also note that this metric on an arbitrary regular Φ-space G/H is naturally reductive with respect to the canonical reductive decomposition [7].

Now we formulate in the most complete form the results concerning the generalized Hermitian geometry of canonical f-structures on homogeneous Φ-spaces of order 4 and 5 with naturally reductive metrics.

Theorem 8. *A canonical metric f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on a naturally reductive homogeneous Φ-space $(G/H, g)$ of order 4 is simultaneously a Hermitian f-structure and a near Kähler f-structure. In addition the following conditions are equivalent*:

1) *f is a quasi-Kähler structure*; 2) *f is a Killing structure*; 3) *f is integrable*; 4) *f is a Kähler structure*; 5) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$; 6) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 7) *$G/H$ is a locally symmetric space.*

Theorem 9. *Let G/H be a naturally reductive Φ-space of order 5, and let f_1, f_2, J_1, J_2 be the canonical structures on G/H. Then each of the structures f_1 and f_2 is a Hermitian f-structure as well as a nearly Kähler f-structure. What is more, the following conditions are equivalent*:

1) *f_1 is a quasi-Kähler structure*; 2) *f_2 is a quasi-Kähler structure*; 3) *f_1 is a Killing structure*; 4) *f_2 is a Killing structure*; 5) *f_1 is integrable*; 6) *f_2 is integrable*; 7) *f_1 is a Kähler structure*; 8) *f_2 is a Kähler structure*; 9) *J_1 and J_2 are NK-structures*; 10) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 11) *$G/H$ is a locally symmetric space.*

Remark 6. The proof of the statements of Theorems 8 and 9 are based on the general results established above and special features of canonical f-structures on homogeneous Φ-spaces of order 4 and 5 [21, 37, 38]. These results were partially announced or proved in [17–21]. A detailed exposition of these and some other related questions will be given in a special paper.

We also note that many concrete examples of homogeneous spaces satisfying the conditions of Theorems 8 and 9 can be found in [39, 40], and some other papers.

In conclusion, note that by now a considerable amount of information has been obtained on canonical f-structures on homogeneous Φ-spaces of order 6, and a series of general facts have been established concerning homogeneous Φ-spaces of arbitrary order k, canonical structures on these spaces, and their relations with generalized Hermitian geometry. In addition, a field of research closely connected with the subject of this paper is now intensively developed aimed at the study of invariant f-structures on flag manifolds (see, e.g., [41]).

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