ISRAEL JOURNAL OF MATHEMATICS 93 (1996), 29–71

REPRESENTATION VARIETIES OF THE FUNDAMENTAL GROUPS OF COMPACT ORIENTABLE SURFACES*

BY

A. S. RAPINCHUK,** V. V. BENYASH-KRIVETZ AND V. I. CHERNOUSOV

Institute of Mathematics, Academy of Sciences of Belarus
ul. Surganova, 11, Minsk, 220072, Belarus
e-mail: imanb%imanb.belpak.minsk.by@demos.su

ABSTRACT

We show that the representation variety for the surface group in characteristic zero is (absolutely) irreducible and rational over $\mathbb{Q}$.

Introduction

Let Γ be a finitely generated group. For any algebraic group G the set $\mathbf{R}(\Gamma, G)$ of all representations (= homomorphisms) $\rho\colon \Gamma \to G$ is known to have a natural structure of an algebraic variety, and endowed with this structure is called **the variety representations of Γ in** G (cf. [Lu-M], [Pl-R]). In the case $G = \mathbf{GL}_n$ which is analyzed by the classical representation theory, $\mathbf{R}(\Gamma,\ \mathbf{GL}_n)$ is denoted simply by $\mathbf{R}_n(\Gamma)$ and called **the variety of n-dimensional representations** of Γ. Since $\mathbf{R}_n(\Gamma)$ is defined by the equations arising from the relations for the generators of Γ, a special role in this theory is played by the one-relator groups

$$\Gamma = \langle x_1, \ldots, x_n \mid r = 1 \rangle.$$

* This work was supported in part by the International Science Foundation (Grant No. MWQ000).

** Visiting the University of Michigan (Ann Arbor, MI 48109, USA) in 1992–94.

Received April 12, 1994

The methods of this paper allow one to consider in full the case: $n \geq 4$, $r = r_1[x_{n-3}, x_{n-2}]\,[x_{n-1}, x_n]$ where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of x and y, and r_1 is an arbitrary word in the derived subgroup of the free group $F(x_1, \ldots, x_{n-4})$. The most notable groups of this kind are the fundamental groups Γ_g of compact orientable surfaces of genus $g > 1$, and that is why we formulate our results (which remain valid also for $g = 1$) for these groups. So, let Γ_g $(g \geq 1)$ be the group with $2g$ generators $x_1, y_1, \ldots, x_g, y_g$ and a single defining relation

$$[x_1, y_1] \cdots [x_g, y_g] = 1.$$

Then a description of $\mathbf{R}_n(\Gamma_g)$ for the ground field of characteristic 0 is given by

THEOREM 1: *$\mathbf{R}_n(\Gamma_g)$ is an (absolutely) irreducible $\mathbb{Q}$-rational variety of dimension*

$$\dim \mathbf{R}_n(\Gamma_g) = \begin{cases} (2g-1)n^2 + 1 & \textit{if}\, g > 1, \\ n^2 + n & \textit{if}\, g = 1. \end{cases}$$

Informally speaking, Theorem 1 means that "almost all" n-dimensional representations of Γ_g can be parameterized by some rational functions thus yielding a nice description of the totality of representations of Γ_g. However to complete in a sense the representation theory for Γ_g one should supplement the latter with a description of the equivalence classes of representations. In geometric terms this amounts to the analysis of the corresponding variety $\mathbf{X}_n(\Gamma_g)$ of **n-dimensional characters**. Recall that $\mathbf{X}_n(\Gamma)$ can be defined as a categorical quotient of $\mathbf{R}_n(\Gamma)$ modulo the action of $\mathbf{GL}_n$ by conjugation and that the points of $\mathbf{X}_n(\Gamma)$ are in one-to-one correspondence with the equivalence classes of fully reducible representations of Γ [Lu-M]. (Another realization of $\mathbf{X}_n(\Gamma)$ is given in [Pl].)

THEOREM 2: *The character variety $\mathbf{X}_n(\Gamma_g)$ is irreducible and $\mathbb{Q}$-unirational, of dimension $(2g-2)n^2 + 2$ (resp., $2n$) for $g > 1$ (resp., $g = 1$). Moreover, $\mathbf{X}_n(\Gamma_g)$ is $\mathbf{Q}$-rational if $g > 1$ and $n \leq 3$.*

Unfortunately, not much more is known about rationality of the character variety even for the apparently simpler case of the free group $\Gamma = F_m$, namely, $\mathbf{X}_n(F_m)$ for $m > 1$ is known to be rational only for $n \leq 4$ (cf. [F1], [F2] and a survey article [LeB]) though it is expected to be rational for n arbitrary (conjecture due to Procesi). Note that a bit weaker property than rationality, that of stable rationality is known to hold for $\mathbf{X}_n(F_m)$ for any $n|$ 420 [B-LeB], however at the moment we do not have any analog of this result for $\mathbf{X}_n(\Gamma_g)$.

The irreducibility of $\mathbf{R}_n(\Gamma_g)$ in Theorem 1 easily implies that of $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)$ (Theorem 3 in §4) and therefore the connectedness of $\mathbf{R}_n(\Gamma_g)_{\mathbb{C}}$ and $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)_{\mathbb{C}}$ [Sh1]. W. Goldman [Go] obtained this result for representations in $\mathbf{SL}_2(\mathbb{C})$ as a by-product of his thorough analysis of the connected components of $\mathbf{R}(\Gamma_g, \mathbf{SL}_2)_{\mathbb{R}}$. He conjectured the connectedness of $\mathbf{R}(\Gamma_g, G)$ for any simple simply connected complex Lie group G. This conjecture was recently proved by Jun Li [J]. However the argument therein does not seem to imply irreducibility even in our case. On the other hand, in some cases you really need to know irreducibility rather than connectedness. For example, as communicated to us by A. Lubotzky, once we know irreducibility of $\mathbf{X}_n(\Gamma_g)$, we can use the method developed in [Ba-Lu] to come up with a very fine approximation of the mapping class group (= Out (Γ_g)). In particular, one can prove it is virtually residually torsion free nilpotent, and consequently, virtually p-residually finite. As a matter of fact, since our approach allows to prove irreducibility of $\mathbf{R}_n(\Gamma)$ for a lot of groups other than Γ_g, we can use [Ba-Lu] to analyze the existence of a similar approximation for some new outer automorphisms groups.

Theorem 1 was initially obtained by the first two authors under two assumptions (cf. a short exposition in [R-Be]). To formulate these we need some notations. Let $T(z)$ for $z \in \mathbf{SL}_n$ denote the Zariski closed subset of all $y \in \mathbf{GL}_n$ such that y and zy have the same characteristic polynomial, and let $W(z) = \{(x, y) \in \mathbf{GL}_n \times \mathbf{GL}_n |\ [x, y] = z\}$ be the corresponding commutator variety (clearly, $T(z)$ contains the projection of $W(z)$ onto the second component). We assumed in [R-Be] that:

(1) *for all z in some Zariski open subset $U \subset \mathbf{SL}_n$ the variety $T(z)$ is irreducible;*
(2) *for any $x, y \in \mathbf{GL}_n$ the coset of x modulo the centralizer of y contains a regular element.*

In [R-Be] we mentioned that both assumptions are valid for $n \leq 4$, and besides, assumption (2) is valid for arbitrary n "generically" (cf. Proposition 2 below). Subsequently, the third author proved the irreducibility of the "generic" commutator variety which is a bit weaker assertion than (1) but still allows one to prove the following statement crucial for the proof of Theorem 1.

PROPOSITION 5: *There exists a $\mathbb{Q}$-defined Zariski open set $U \subset \mathbf{SL}_n$ such that for any extension $K/\mathbb{Q}$ and for any $z \in U_K$ the commutator variety $W(z)$ is an* *(absolutely) irreducible K-rational variety of dimension* (n^2+1).

The third author also proved assumption (2) in full. (Note that when this work was done D. Djokovič kindly sent us his preprint [D] in which he also proved this assertion.)

It would be interesting to extend Theorem 1 to the representations of Γ_g into other algebraic groups. Recently the first and the third authors proved that the variety $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)$ is irreducible and $\mathbb{Q}$-unirational (Theorem 3). The proof of irreducibility here actually does not differ from the case of the group $\mathbf{GL}_n$; however the proof of unirationality uses a new idea which we are going to explain now. If we are given two elements $a, b \in \mathbf{GL}_n$, and multiply a by an element of the centralizer of b or, symmetrically, multiply b by an element of the centralizer of a (i.e. perform a so-called standard transformation) we do not alter the commutator $[a,b]$. Now suppose one is given $a,b,c,d \in \mathbf{GL}_n$ (resp., $\mathbf{SL}_n$) such that $[a,b] = [c,d]$, one may ask if it is possible to pass from (a,b) to (c,d) by a chain of standard transformations. We prove that this is indeed true "generically", i.e. for (a,b,c,d) in some Zariski open subset of $(\mathbf{GL}_n)^4$ (Theorem 4), and this fact allows us to prove the unirationality of the generic commutator variety in $\mathbf{SL}_n$ and eventually of $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)$.

The main results of this paper were presented at the Norman–Minsk workshop "Representation Varieties and Platonov's Conjecture" organized by the University of Oklahoma (Norman, Ok. USA, the NSF grant INT-9206790) whose hospitality is gratefully acknowledged. We are grateful to the participants of the workshop, especially to A. Magid and A. Lubotzky, for helpful discussions.

1. Some results on regular elements

Recall that an element x of a reductive algebraic group G is called **regular** if its centralizer $Z_G(x)$ has minimal possible dimension (which is always equal to the rank of G). It is well-known that the set G_{reg} of regular elements is Zariski open in G [Sp-St]. Henceforth we shall be working with regular elements only in the group $G = \mathbf{GL}_n$, and therefore $Z_G(x)$ will be denoted simply as $Z(x)$. One can easily check that in this case an element $x \in G$ is regular if and only if in its Jordan normal form every eigenvalue corresponds to a single Jordan block. Note that the latter condition can be reformulated as follows: for any λ in the algebraic closure of the field of definition, the rank of the matrix $x - \lambda E_n$ is $\geq n-1$ (E_n is the unit matrix); in particular, a semisimple $x \in G$ is regular iff each of its eigenvalues has multiplicity one. We need to know that regular elements can be found among the elements of a specific form.

PROPOSITION 1: *For any $x, y \in \mathbf{GL}_n$ the set $xZ(y)$ contains a regular element.*

Proof: Replacing x and y by suitable conjugates we may (and we will) assume y to be a Jordan matrix, i.e. $y = \operatorname{diag}(J_{k_1}(\alpha_1), \dots, J_{k_m}(\alpha_m))$ where

$$J_k(\alpha) = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \dots & 0 \\ 0 & 0 & \alpha & \dots & 0 \\ & & & \ddots & \\ 0 & & \dots & & \alpha \end{pmatrix}$$

is a Jordan block of order $k, \alpha, \alpha_1, \dots, \alpha_m$ belong to the fixed algebraically closed field K. It is easy to check that the centralizer of $J_k(\alpha)$ in the full matrix algebra $\mathbf{M}_k(K)$ consists of all matrices of the form

$$X_k(a_1, \dots, a_k) = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ 0 & a_1 & \dots & a_{k-1} \\ & & \ddots & \\ 0 & 0 & \dots & a_1 \end{pmatrix} \qquad (a_i \in K).$$

Now let $x_1^1, \dots, x_{k_1}^1, \dots, x_1^m, \dots, x_{k_m}^m$ be algebraically independent over K and

$$c(k_1, \dots, k_m) = \operatorname{diag}(X_{k_1}(x_1^1, \dots, x_{k_1}^1), \dots, X_{k_m}(x_1^m, \dots, x_{k_m}^m)).$$

Then $c = c(k_1, \dots, k_m)$ is a non-degenerate matrix over the field

$$L = K(x_1^1, \dots, x_{k_1}^1, \dots, x_1^m, \dots, x_{k_m}^m)$$

commuting with y. Besides, since the set of regular elements is open in the Zariski K-topology it suffices to show that xc^{-1} is regular. Thus, we need to prove that for any λ in the algebraic closure $\bar{L}$ we have the following inequality for the ranks of matrices:

$$\operatorname{rk}(xc^{-1} - \lambda E_n) = \operatorname{rk}(x - \lambda c) \geq n - 1.$$

It turns out that the latter fact is true for arbitrary $x \in \mathbf{M}_n(K)$ (even degenerate).

LEMMA 1: *Let $x \in \mathbf{M}_n(K)$, c be as introduced above. Then for any $\lambda \in \bar{L}$ we have* $\operatorname{rk}(x - \lambda c) \geq n - 1$.

Proof: Put

$$F_n(k_1,\ldots,k_m)=\{x-\lambda c(k_1,\ldots,k_m)|\ x\in\mathbf{M}_n(K),\ \lambda\in\bar{L}\}.$$

We are going to show by induction on n that for any choice of $k_1,\ldots,k_m$ the rank of any element from $F_n(k_1,\ldots,k_m)$ is $\geq n-1$. For $n=2$ this is easily verified by direct computations. Assume that for some $n>2$, there are some $k_1,\ldots,k_m$ such that $k_1+\cdots+k_m=n$ and the rank of $b=x-\lambda c\in F_n(k_1,\ldots,k_m)$ is strictly less than $n-1$. Let x_1 (resp., b_1) be the matrix obtained from x (resp., b) by deleting the last row and the last column. Then

$$b_1=x_1-\lambda c_1$$

where $c_1=\operatorname{diag}(X_{k_1}(x_1^1,\ldots,x_{k_1}^1),\ldots,\ X_{k_m-1}(x_1^m,\ldots,x_{k_m-1}^m))$. Since $\operatorname{rk} b_1\leq \operatorname{rk} b<n-1$, we have $\det b_1=(\det c_1)(\det(x_1c_1^{-1}-\lambda E_{n-1}))=0$. Hence λ is an eigenvalue of the matrix $x_1c_1^{-1}$, in particular, λ belongs to the algebraic closure of the field $L_1=K(x_1^1,\ldots,x_{k_m-1}^m)$, i.e. λ is independent of the last variable $x_{k_m}^m$. Then $b_1\in F_{n-1}(k_1,\ldots,k_m-1)$ (under the convention that if $k_m=1$ we put $F_{n-1}(k_1,\ldots,k_{m-1},0)=F_{n-1}(k_1,\ldots,k_{m-1})$), and by inductive hypothesis $\operatorname{rk} b_1\geq n-2$. Consequently, $\operatorname{rk} b=\operatorname{rk} b_1=n-2$. Moreover, if $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are the columns of b then a base of the vector subspace $\langle\mathbf{e}_1,\ldots,\mathbf{e}_n\rangle\subset\bar{L}^n$ generated over $\bar{L}$ by $\mathbf{e}_1,\ldots,\mathbf{e}_n$ can be chosen from among $\mathbf{e}_1,\ldots,\mathbf{e}_{n-1}$, i.e. the vector $\mathbf{e}_n$ is in fact a linear combination of the others.

Now consider the projection $\bar{L}^n\to\bar{L}^{n-1}$, $v\mapsto\tilde{v}$, omitting the $(n-k_m+1)$-th component. We claim that

$$\dim\langle\tilde{\mathbf{e}}_1,\ldots,\tilde{\mathbf{e}}_n\rangle=n-2. \tag{1}$$

Indeed, if x_2 (resp., b_2) is the matrix obtained from x (resp., b) by deleting the row and the column with the number $(n-k_m+1)$, then clearly

$$b_2=x_2-\lambda c_1$$

with the same c_1 as above, implying that $b_2\in F_{n-1}(k_1,\ldots,k_m-1)$. Arguing as above we obtain that in our setting $\operatorname{rk} b=\operatorname{rk} b_2=n-2$ yielding (1). Since $\mathbf{e}_n$ is a linear combination of $\mathbf{e}_1,\ldots,\mathbf{e}_{n-1}$, then $\tilde{\mathbf{e}}_n$ is a linear combination of $\tilde{\mathbf{e}}_1,\ldots,\tilde{\mathbf{e}}_{n-1}$, so one can find a base $\tilde{\mathbf{e}}_{i_1},\ldots,\tilde{\mathbf{e}}_{i_{n-2}}$ of the space $\langle\tilde{\mathbf{e}}_1,\ldots,\tilde{\mathbf{e}}_n\rangle$ such that all i_j's are different from n. The corresponding vectors $\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_{n-2}}$ are linearly independent and therefore form a base of the space $\langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle$. Consider a presentation

$$\mathbf{e}_n = \beta_1 \mathbf{e}_{i_1} + \cdots + \beta_{n-2} \mathbf{e}_{i_{n-2}}, \quad \beta_i \in \bar{L}. \tag{2}$$

We claim that all β_i in (2) in fact belong to $\bar{L}_1$. Indeed, from (2) one obtains

$$\tilde{\mathbf{e}}_n = \beta_1 \tilde{\mathbf{e}}_{i_1} + \cdots + \beta_{n-2} \tilde{\mathbf{e}}_{i_{n-2}}.$$

However all the coordinates of $\tilde{\mathbf{e}}_n, \tilde{\mathbf{e}}_{i_1}, \ldots, \tilde{\mathbf{e}}_{i_{n-2}}$ belong to $\bar{L}_1$, consequently all β_i belong to $\bar{L}_1$ too, as required. Now matching the $(n - k_m + 1)$-th coordinates in (2) and bearing in mind that these coordinates of all of the $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-2}}$ are in $\bar{L}_1$ we derive that $x^m_{k_m} \in \bar{L}_1$. The contradiction proves Proposition 1. ∎

COROLLARY: *Let $z \in \mathbf{SL}_n$, $W(z) = \{(x, y) \in \mathbf{GL}_n \times \mathbf{GL}_n \mid [x, y] = z\}$ be the corresponding commutator variety. Then any irreducible component $W_1 \subset W(z)$ contains a point (x, y) such that both x and y are regular elements.*

Proof: Let $(x_1, y_1) \in W_1$ be a point which does not belong to any other irreducible component of $W(z)$.

The set $C = (x_1 Z(y_1), y_1)$ is irreducible (indeed, $Z(y_1)$ is the Zariski open subset of $Z_{M_n}(y_1)$, the centralizer of y_1 in the matrix algebra, which is the affine space) and contained in $W(z)$, so it must lie entirely inside some irreducible component. However due to our choice of (x_1, y_1), this component cannot be other but W_1, i.e. $C \subset W_1$. Now, applying Proposition 1 and using irreducibility of $Z(y_1)$ we can pick a regular element $x \in x_1 Z(y_1)$ such that (x, y_1) belongs to W_1 but not to any other irreducible component of $W(z)$. Repeating this argument we can find $(x, y) \in W_1$ with both x and y regular. ∎

For the case of semisimple y and char $K = 0$ one can prove the following refinement of Proposition 1.

PROPOSITION 2 ([R-Be]): *Let* $\operatorname{char} K = 0$. *If $x, y \in \mathbf{GL}_n$ and y is semisimple, then the set $xZ(y)$ contains a regular semisimple element.*

Proof: Conjugating y we may assume it to be diagonal, and then we have in fact to prove that given $x \in \mathbf{GL}_n$ there exists a diagonal matrix $d \in \mathbf{D}_n$ such that the matrix xd has distinct eigenvalues. For a polynomial $p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ let $\Delta_n = \Delta_n(a_1, \ldots, a_n)$ denote the resultant of the polynomials $p(\lambda)$ and $p'(\lambda)$

which, up to the sign, coincides with the discriminant of $p(\lambda)$. We need the following lemma.

LEMMA 2:
(i) $\Delta_n \Big|_{a_n=0} = (-1)^{n-1}a_{n-1}^2\Delta_{n-1}(a_1,\dots,a_{n-1})$.
(ii) *For $n > k \geq 1$ we have*

$$\Delta_n \Big|_{a_{k+1}=a_{k+2}=\cdots=a_{n-1}=0} = (-1)^{nk}a_n^{n-k-1}\tilde{\Delta}_n$$

for some polynomial $\tilde{\Delta}_n$ in $a_1,\dots,a_k,a_n$ such that

$$\tilde{\Delta}_n \Big|_{a_n=0} = (-1)^k(n-k)^{n-k}a_k^{n-k+1}\Delta_k(a_1,\dots,a_k)$$

(here we assume that $\Delta_1 := 1$).

Proof: (i) It follows from the definition of the resultant that

$$\Delta_n \Big|_{a_n=0} = \begin{vmatrix} 1 & a_1 & \dots & a_{n-1} & 0 & & & \\ & \ddots & & & & & & \\ & & & 1 & a_1 & \dots & a_{n-1} & 0 \\ n & (n-1)a_1 & \dots & a_{n-1} & & & & \\ & & \ddots & & & & & \\ & & & n & (n-1)a_1 & \dots & a_{n-1} & \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n-1 \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n \text{ rows} \end{array}$$

The only non-zero entry in the last column of this determinant is a_{n-1}, so expanding it along the last column we get

$$\Delta_n \Big|_{a_n=0} = a_{n-1} \begin{vmatrix} 1 & a_1 & \dots & a_{n-1} & & & \\ & \ddots & & & & & \\ & & 1 & a_1 & \dots & a_{n-1} & \\ n & (n-1)a_1 & \dots & a_{n-1} & & & \\ & \ddots & & & & & \\ & & n & (n-1)a_1 & \dots & a_{n-1} & \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n-1 \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n-1 \text{ rows} \end{array}$$

Now, let us make the following transformations of the latter determinant: for each $i = 0,\dots,n-2$ we subtract the $(i+1)$-th row from the $(n+i)$-th row. Then we get a determinant with the only non-zero element in the last column, this element in the position $(n-1, 2n-2)$ and being equal to a_{n-1}. Expanding this determinant along the last column we get $\Delta_n \Big|_{a_n=0} =$

$$a_{n-1}(-1)^{n-1}a_{n-1}\begin{vmatrix} 1 & a_1 & \dots & a_{n-1} & & & \\ & \ddots & & & & & \\ & & & 1 & a_1 & \dots & a_{n-1} \\ n-1 & (n-2)a_1 & \dots & a_{n-2} & & & \\ & \ddots & & & & & \\ & & & n-1 & (n-2)a_1 & \dots & a_{n-2} \end{vmatrix}$$

$= (-1)^{n-1}a_{n-1}^2\Delta_{n-1}(a_1,\dots,a_{n-1})$, as required.

(ii) Again, by the definition of the resultant

$$\Delta_n \Big|_{a_{k+1}=\dots=a_{n-1}=0} =$$

$$\begin{vmatrix} 1 & a_1 & \dots & a_k & 0 & \dots & 0 & a_n & & \\ & \ddots & & & & & & & & \\ & & 1 & a_1 & \dots & a_k & 0 & \dots & 0 & a_n \\ n & (n-1)a_1 & & \dots & & (n-k)a_k & & & & \\ & \ddots & & & & & & & & \\ & & n & (n-1)a_1 & & \dots & & (n-k)a_k & & \end{vmatrix}$$

(the size of the matrix is $(n-1)+n$). Noticing that this determinant looks like $\begin{vmatrix} * & 0 \\ * & a_nE_{n-k-1} \\ * & 0 \end{vmatrix}$ and expanding it along the last $(n-k-1)$ columns, we obtain

$$\Delta_n \Big|_{a_{k+1}=\dots=a_{n-1}=0} = (-1)^{n(n-k-1)}a_n^{n-k-1}\tilde{\Delta}_n = (-1)^{nk}a_n^{n-k-1}\tilde{\Delta}_n$$

with the obvious $\tilde{\Delta}_n$. Furthermore, it is easy to see that

$$\tilde{\Delta}_n \Big|_{a_n=0} = \begin{vmatrix} 1 & a_1 & \dots & a_k & & & \\ & \ddots & & & & & \\ & & 1 & a_1 & & \dots & a_k \\ n & (n-1)a_1 & \dots & (n-k)a_k & & & \\ & \ddots & & & & & \\ & & n & (n-1)a_1 & \dots & & (n-k)a_k \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \begin{matrix} k \\ \text{rows} \end{matrix} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \begin{matrix} n \\ \text{rows} \end{matrix} \end{array}$$

This determinant is of the form $\begin{vmatrix} * & 0 \\ * & (n-k)a_kE_{n-k} \end{vmatrix}$ so $\tilde{\Delta}_n \Big|_{a_n=0} =$

$$(n-k)^{n-k}a_k^{n-k}\begin{vmatrix} 1 & a_1 & \dots & a_k & & & \\ & \ddots & & & & & \\ & & 1 & a_1 & \dots & a_k \\ n & (n-1)a_1 & \dots & (n-k)a_k & & & \\ & \ddots & & & & & \\ & & n & (n-1)a_1 & \dots & (n-k)a_k \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ rows} \end{array}$$

In the latter determinant we make the following transformations: for each $i = 1, \dots, k$ we subtract from the $(k+i)$-th row the i-th row multiplied by $(n-k)$. Then we get

$$\tilde{\Delta}_n \Big|_{a_n=0} =$$

$$(n-k)^{n-k}a_k^{n-k}\begin{vmatrix} 1 & a_1 & \dots & a_k & & & \\ & \ddots & & & & & \\ & & 1 & a_1 & \dots & a_k \\ k & (k-1)a_1 & \dots & a_{k-1} & & & \\ & \ddots & & & & & \\ & & k & (k-1)a_1 & \dots & a_{k-1} \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ rows} \end{array}$$

Finally, expanding the obtained determinant along the last column we arrive at the required formula

$$\tilde{\Delta}_n \Big|_{a_n=0} = (-1)^n(n-k)^{n-k}a_k^{n-k+1}\Delta_k(a_1, \dots, a_k).$$

Lemma 2 is proved. ∎

For $z \in \mathbf{GL}_n$ let $f_z(\lambda) = \det(\lambda E_n - z)$ be the characteristic polynomial of z, $f_z(\lambda) = \lambda^n + \sigma_1(z)\lambda^{n-1} + \dots + \sigma_n(z)$, and let $\Delta_n(z) = \Delta_n(\sigma_1(z), \dots, \sigma_n(z))$. For the proof of Proposition 2 it suffices to find $d \in \mathbf{D}_n$ such that $\Delta_n(xd) \neq 0$. Let $d_1, \dots, d_n$ be algebraically independent over K and let $d = \mathrm{diag}(d_1, \dots, d_n)$ be a "generic" diagonal matrix. We are going to show that $\Delta_n(xd)$ as a polynomial in $d_1, \dots, d_n$ is not identically zero. Let us proceed by induction on n. The case $n = 2$ can be easily handled by direct computations. Now suppose $n > 2$ and our assertion has been proved already for all $k < n$. Let $M_{j_1,\dots,j_s}$ denote the principal minor of the matrix x located in rows and columns with the numbers $j_1, \dots, j_s$.

It is easy to see that the coefficients $\sigma_i(x)$ of the characteristic polynomial $f_x(\lambda)$ are expressed as follows:

$$\sigma_i(x) = (-1)^i \sum_{j_1<\cdots<j_s} M_{j_1,\ldots,j_s}.$$

Therefore,

$$\sigma_i(xd) = (-1)^i \sum_{j_1<\cdots<j_s} d_{j_1}\cdots d_{j_s} M_{j_1,\ldots,j_s}.$$

To begin with, assume that $\sigma_{n-1}(xd) \not\equiv 0$. This means that not all principal minors of order $(n-1)$ of x vanish, and in this case after conjugating x by a suitable monomial matrix (that induces a permutation of the diagonal entries of d) we may (and we will) assume that $M_{1,\ldots,n-1} \neq 0$. Let x' denote the matrix obtained from x by deleting the n-th row and the n-th column (so that $M_{1,\ldots,n-1}$ $= \det x'$). Also put $d^{(1)} = \operatorname{diag}(d_1,\ldots,d_{n-1},0), d' = \operatorname{diag}(d_1,\ldots,d_{n-1})$. Then as one can easily see we have

$$\sigma_i(xd^{(1)}) = \sigma_i(x'd') \quad \text{for } i \leq n-1,$$

and

$$\sigma_n(xd^{(1)}) = 0.$$

Using (i) of Lemma 2 we obtain

$$\Delta_n(xd)\,\Big|_{d_n=0} = \Delta_n(xd^{(1)}) = (-1)^{n-1}\sigma_{n-1}(xd^{(1)})^2\Delta_{n-1}(x'd').$$

By induction hypothesis, $\Delta_{n-1}(x'd') \not\equiv 0$. On the other hand,

$$\sigma_{n-1}(xd^{(1)}) = (-1)^{n-1} d_1\cdots d_{n-1} M_{1,\ldots,n-1} \not\equiv 0.$$

Thus, $\Delta_n(xd) \not\equiv 0$, as desired.

Now let $\sigma_{n-1}(xd) \equiv 0$. If $\sigma_1(xd) = \cdots = \sigma_{n-1}(xd) \equiv 0$, then $f_{xd}(\lambda) = \lambda^n + \sigma_n(xd)$ has no multiple roots implying $\Delta_n(xd) \not\equiv 0$. So, we may assume that for some k, $1 \leq k < n-1$, we have

$$\sigma_{k+1}(xd) = \cdots = \sigma_{n-1}(xd) \equiv 0,$$

but

$$\sigma_k(xd) \not\equiv 0.$$

Then using notation introduced in part (ii) of Lemma 2 and putting $\tilde{\Delta}_n(xd) = \tilde{\Delta}_n(\sigma_1(xd), \ldots, \sigma_n(xd))$ we may write

$$\Delta_n(xd) = (-1)^{nk}\sigma_n(xd)^{n-k-1}\tilde{\Delta}_n(xd).$$

Since $\sigma_n(xd) = (-1)^n \det(xd) \not\equiv 0$, it suffices to show that $\tilde{\Delta}_n(xd) \not\equiv 0$. But by assumption $\sigma_k(xd) \not\equiv 0$, hence not all the principal minors of order k of x vanish, and just as above we may assume that $M_{1,\ldots,k} \neq 0$. Let x'' be the matrix obtained from x by deleting rows and columns with the numbers $k+1, \ldots, n$. Put

$$d^{(2)} = \operatorname{diag}(d_1, \ldots, d_k, 0, \ldots, 0), \quad d'' = \operatorname{diag}(d_1, \ldots, d_k).$$

Then it is easy to see that

$$\sigma_i(xd^{(2)}) = \begin{cases} \sigma_i(x''d'') & \text{for } i \leq k, \\ 0 & \text{for } i > k. \end{cases}$$

By virtue of Lemma 2 (ii) we have

$$\begin{aligned} \tilde{\Delta}_n(xd) \Big|_{d_{k+1}=\cdots=d_n=0} &= \tilde{\Delta}_n(xd^{(2)}) \\ &= (-1)^k(n-k)^{n-k}\sigma_k(xd^{(2)})^{n-k+1}\Delta_k(x''d''). \end{aligned}$$

However, $\sigma_k(xd^{(2)}) = (-1)^k d_1 \cdots d_k \ M_{1,\ldots,k} \not\equiv 0$ and $\Delta_k(x''d'') \not\equiv 0$ by the inductive hypothesis, hence $\tilde{\Delta}_n(xd) \not\equiv 0$. Proposition 2 is proved. ∎

Remark: As observed by V. P. Platonov, the analog of Proposition 1 for an arbitrary reductive group is false even if the group is simple and the element y is semisimple.

2. Commutator varieties in $\mathbf{GL}_n$

First, we consider the case $g = 1$. Here $\Gamma = \langle x, y \mid [x, y] = 1\rangle$, i.e. Γ is a free abelian group on two generators. Then $\mathbf{R}_n(\Gamma)$ coincides with the variety $\mathbf{C}(2, n)$ of pairs of commuting matrices in $\mathbf{GL}_n$. More generally, we define the variety $\mathbf{C}(p, n)$ of p-tuples of pairwise commuting $(n \times n)$-matrices to be $\{(x_1, \ldots, x_p) \in (\mathbf{GL}_n)^p \mid x_i x_j = x_j x_i \text{ for all } i, j = 1, \ldots, p\}$, and then $\mathbf{C}(p, n) = \mathbf{R}_n(\mathbf{Z}^p)$. The irreducibility of $\mathbf{C}(2, n)$ was established by Motzkin and Taussky [Mo-Ta] (cf. also [G]), and afterwards this result has been generalized by Richardson [Ri] to the variety $\mathbf{C}(2, G)$ of commuting elements in an arbitrary reductive group

G.* For the sake of completeness we give a simple proof of irreducibility (and rationality) of $\mathbf{R}_n(\Gamma)$ based on the following elementary lemma which will also be used in the sequel.

LEMMA 3: *Let U be an irreducible K-defined algebraic variety. Fix an integer $n > 0$ and consider a subvariety $X \subset U \times \mathbb{A}^n$ defined by a system of linear equations:*

$$\sum_{j=1}^{n} f_{ij}(u)t_j = g_i(u), \quad i = 1, \ldots, m,$$

where $f_{ij}(u), g_i(u) \in K[U]$ are regular functions and $t_1, \ldots, t_n$ are the coordinates in $\mathbb{A}^n$. Put

$$F(u) = \begin{pmatrix} f_{11}(u) & \ldots & f_{1n}(u) \\ \vdots & & \\ f_{m1}(u) & \ldots & f_{mn}(u) \end{pmatrix},$$

$$\tilde{F}(u) = \begin{pmatrix} f_{11}(u) & \ldots & f_{1n}(u) & g_1(u) \\ \vdots & & & \\ f_{m1}(u) & \ldots & f_{mn}(u) & g_m(u) \end{pmatrix}$$

and assume that everywhere on U we have $\operatorname{rk} F(u) = \operatorname{rk} \tilde{F}(u) = r$ for some constant r. Then X is irreducible and the field of rational functions $K(X)$ is isomorphic to $K(U)(s_1, \ldots, s_{n-r})$ for some algebraically independent parameters $s_1, \ldots, s_{n-r}$.

Proof: Let $m_1(u), \ldots, m_l(u)$ be all $(r \times r)$-minors of $F(u)$. Put $U_i = \{u \in U |\ m_i(u) \neq 0\}, X_i = \{(u, a) \in X |\ u \in U_i\}$ and $I = \{i |\ U_i \neq \emptyset\}$. It follows from Cramer's Rule for linear systems and our assumptions that for every $i \in I$ we have $X_i \simeq U_i \times \mathbf{A}^{n-r}$. In particular, X_i is an irreducible variety of dimension $d = \dim U + n - r$. Since U is irreducible, for any $i, j \in I$, the intersection $X_i \cap X_j$ is non-empty, and therefore is, in fact, dense in both X_i and X_j. Besides, $X = \bigcup_{i \in I} X_i$ implying, in particular, that $\dim X = d$. Fix some $i \in I$ and let X' be an irreducible component of X containing X_i. If we assume that X is reducible then for some $j \in I$ we shall have $X_j \not\subset X'$. Let X'' be a component containing X_j. Since $\dim X' = \dim X_i = d, X_i$ is dense in X'. For the same reason, X_j is dense in X''. Therefore, $X_i \cap X_j$ is dense in both X' and X'', forcing $X' = X''$. The contradiction proves the irreducibility of X. Now if

* The first author wishes to thank J. T. Stafford for pointing out to him references [G] and [Ri].

we fix some $i \in I$, then $K(X) = K(X_i) = K(U_i \times \mathbf{A}^{n-r}) = K(U_i)(s_1, \ldots, s_{n-r})$ $= K(U)(s_1, \ldots, s_{n-r})$ as required. Lemma 3 is proved. ∎

PROPOSITION 3: *For $\Gamma = \mathbb{Z}^2$, $\mathbf{R}_n(\Gamma) = \mathbf{C}(2,n)$ is an (absolutely) irreducible $\mathbf{Q}$-rational variety.*

Proof: Let $U \subset \mathbf{GL}_n$ be the set of regular elements. Consider the following open subset $X \subset \mathbf{C}(2,n)$:

$$X = \{(x,y) \in \mathbf{C}(2,n) |\, x \in U\}.$$

It clearly follows from Lemma 3 that X is irreducible and rational over $\mathbb{Q}$. Repeating verbatim the arguments used in proving the Corollary in §1, we see that any component $C' \subset \mathbf{C}(2,n)$ has to meet X. Then $X \cap C'$ is dense in C', so the irreducible component $C_0 \subset \mathbf{C}(2,n)$ containing X contains, in fact, any other component implying the irreducibility of $\mathbf{C}(2,n)$.

Proposition 3 is proved. ∎

In his paper [G] Gerstenhaber observed that $\mathbf{C}(p,2)$ and $\mathbf{C}(p,3)$ are irreducible varieties for p arbitrary (for $\mathbf{C}(p,2)$ this easily follows from Lemma 3 since for any $(p-1)$ pairwise commuting matrices $x_1, \ldots, x_{p-1} \in \mathbf{GL}_2$ their common centralizer $Z = Z(x_1, \ldots, x_{p-1})$ contains a regular element, this fact being false for all $n, p \geq 3$), however for any $n \geq 4$ and any $p \geq n+1$ the variety $\mathbf{C}(p,n)$ is reducible. This result prompted him to ask whether $\mathbf{C}(p,n)$ is irreducible for $n \geq 4$ and $2 < p < n+1$. In the preliminary version of this paper [R-Be-Ch] we gave a proof of the reducibility of $\mathbf{C}(p,n)$ for $n \geq 4$ and $p \geq 4$. However later V. P. Platonov has drawn our attention to the paper of Guralnick [Gu] in which this result had already been obtained. Moreover, Guralnick proved that $\mathbf{C}(3,n)$ is reducible for $n \geq 32$. With slight modification of his argument, one can improve the lower bound to $n \geq 29$. The question on reducibility of $\mathbf{C}(3,n)$ for $4 \leq n \leq 29$ is currently open.

In the remaining part of this section $g > 1$. For $z \in \mathbf{SL}_n$ let $W(z) = \{(x,y) \in \mathbf{GL}_n \times \mathbf{GL}_n |\, [x,y] = z\}$ denote the corresponding commutator variety. It has been known for a long time that for an (infinite) field K any matrix in $\mathbf{SL}_n(K)$ is a commutator of two matrices from $\mathbf{GL}_n(K)$ (cf. [T], note that actually a much sharper result was proved in [T], viz. any noncentral matrix in $\mathbf{SL}_n(K)$ is a commutator already in $\mathbf{SL}_n(K)$). This means that for $z \in \mathbf{SL}_n(K)$

we always $W(z)_K \neq \emptyset$. The proposition below shows that "generically" $W(z)_K$ admits a nice geometric description.

PROPOSITION 4: *There exists a $\mathbb{Q}$-defined Zariski open set $U \subset \mathbf{SL}_n$ such that for any extension $K/\mathbb{Q}$ and any point $z \in U_K$ the commutator variety $W(z)$ is an (absolutely) irreducible K-rational variety of dimension (n^2+1).*

For any matrix $a \in \mathbf{M}_n$ let $f_a(\lambda) = \det\ (\lambda E_n - a)$ be the characteristic polynomial of a, and let $\sigma_1(a), \dots, \sigma_n(a)$ be its coefficients so that

$$f_a(\lambda) = \lambda^n + \sigma_1(a)\lambda^{n-1} + \cdots + \sigma_n(a).$$

Let us consider the following varieties:

$$\begin{aligned} T =& \{(y,z) \in \mathbf{GL}_n \times \mathbf{SL}_n \mid \sigma_i(zy) = \sigma_i(y), i = 1, \dots, n-1\}, \\ S =& \mathbb{A}^{n^2-n} \times \mathbf{SL}_n, \end{aligned}$$

and introduce the following morphisms:

$$\begin{aligned} &\phi\colon \mathbf{GL}_n \times \mathbf{GL}_n \to \mathbf{SL}_n, \quad (x,y) \mapsto [x,y], \\ &\psi\colon \mathbf{GL}_n \times \mathbf{GL}_n \to T, \quad (x,y) \mapsto (y, [x,y]), \\ &\pi\colon T \to S, \quad ((y_{ij})_{1\le i,j\le n}, z) \mapsto ((y_{ij})_{\substack{1\le i\le n \\ 2\le j\le n}}, z). \end{aligned}$$

To prove Proposition 4 we first prove the following weaker assertion.

PROPOSITION 5: *There exists $\mathbb{Q}$-defined Zariski open sets $V \subset \mathbf{GL}_n \times \mathbf{GL}_n$, $U \subset \mathbf{SL}_n$ such that for any extension $K/\mathbb{Q}$ and any point $z \in U_K$ the variety $\phi^{-1}(z) \cap V$ is irreducible and rational over K.*

(Note that $\phi^{-1}(z)$ coincides with the commutator variety $W(z)$).

Proof: To begin with, let us analyze the system determining T. It follows from the characteristic polynomial $f_a(\lambda)$ of a matrix $a = (a_{ij})$ that its coefficient $\sigma_t(a)$ at λ^{n-t} is equal (up to sign) to the sum of all principal minors of a of order t. Expanding those of them which contain a_{11} along the first column, we obtain for $\sigma_t(a)$ an expression of the form:

$$\sigma_t(a) = \sum_{s=1}^{n} P_{st} a_{s1} + Q_t$$

for some polynomials $P_{st}, Q_t \in \mathbb{Q}[a_{ij}]_{\substack{1\le i\le n \\ 2\le j\le n}}$. Then

$$\sigma_t(y) = \sum_{s=1}^{n} p_{st} y_{s1} + q_t$$

where p_{st}, q_t are obtained from P_{st}, Q_t replacing a_{ij} with y_{ij}, and

$$\sigma_t(zy) = \sum_{s=1}^{n} p'_{st} y'_{s1} + q'_t$$

where y'_{ij} stands for (ij)-entry of the matrix zy and p'_{st}, q'_t are obtained from P_{st}, Q_t by replacing a_{ij} with y'_{ij}. Now, note that y'_{ij} for $j \geq 2$ depends only on the coefficients of z and y_{lk} with $k \geq 2$. So,

$$\sigma_t(zy) = \sum_{s=1}^{n} p''_{st} y_{s1} + q''_t \tag{3}$$

for some $p''_{st}, q''_t \in \mathbf{Q}[\mathbf{SL}_n][y_{ij}]_{\substack{1 \leq i \leq n \\ 2 \leq j \leq n}} = \mathbf{Q}[S]$. Eventually, the system determining T reduces to a system of $(n-1)$ linear equations for the entries of the first column of the matrix y:

$$\sum_{s=1}^{n} c_{st} y_{s1} = d_t, \quad t = 1, \ldots, n-1 \tag{4}$$

for some $c_{st}, d_t \in \mathbb{Q}[S]$. Let us introduce a $\mathbb{Q}$-defined Zariski open set $T_0 \subset T$ consisting of points (y, z) subject to the following conditions:

(i) y is regular and semisimple;

(ii) the rank of the matrix $C(\pi(y,z)) = (c_{ij}(\pi(y,z)))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-1}}$ is $(n-1)$.

Now, we need to make sure that $T_0 \neq \emptyset$. Let $S_0 \subset S$ denote the set of points (a, z) such that $\operatorname{rk} C(a,z) = n-1$. We'll construct a specific point $(a,z) \in S_0$ which can be lifted to a point $(y,z) \in T_0$.

Take $z = \operatorname{diag}(\rho, \ldots, \rho)$ where ρ is a primitive n-th root of unity. Then for any y we have $\sigma_t(zy) = \rho^t \sigma_t(y)$, so the system $\sigma_t(zy) = \sigma_t(y)$ up to the constants coincides with the system

$$\sigma_i(y) = 0, \quad i = 1, \ldots, n-1 \tag{5}$$

Now, let $a = (a_{ij})_{\substack{1 \leq i \leq n \\ 2 \leq j \leq n}} \in \mathbb{A}^{n^2-n}$ be the point with coordinates $a_{ij} = \delta_{ij-1}$ (Kronecker delta). Put $y_{ij} = a_{ij}$ for $i = 1, \ldots, n$; $j = 2, \ldots, n$, and let $y_{11}, \ldots, y_{n1}$ be indeterminates. We claim that the system (5) which is equivalent to the system (4) is then triangular:

$$\left\{ \begin{array}{lcl} y_{11} & = & * \\ *y_{11} - y_{21} & = & * \\ \ldots\ldots\ldots\ldots\ldots & & \\ *y_{11} + *y_{21} + \cdots + (-1)^n y_{n-11} & = & * \end{array} \right. \tag{6}$$

and therefore has rank $n-1$. To prove this, it suffices to show that any principal minor of y containing y_{11}, and different from the minor in the left upper corner, is zero. But it is easy to see that if we pick a minor lying in the rows and columns with numbers $1, 2, \ldots, l, m, \ldots, r$ and $m > l+1$ then its $(l+1)$-th column consists entirely of zeros, and our assertion is evident. Solving (6) for $y_{11}, \ldots, y_{n-1,1}$ we then make our choice of y_{n1} such that for $y = (y_{ij})_{1 \le i,j \le n}$ we have $\det y \neq 0$, which is possible since

$$\det y = *y_{11} + \cdots + *y_{n-1,1} + (-1)^{n+1} y_{n1}. \tag{7}$$

By our construction the characteristic polynomial of y is $f_y(\lambda) = \lambda^n + (-1)^n \det y$ implying that y is regular and semisimple, and so $(y, z) \in T_0$. Let us point out the following by-product of the argument above: We have shown that $S_0 \neq \emptyset$; on the other hand, it follows from the Cramer's Rule for linear systems that $S_0 \subset \operatorname{Im} \pi$ implying that $\pi\colon T \to S$ is dominant.

Let $S_1 \subset S$ be a $\mathbb{Q}$-defined Zariski open subset contained in $\pi(T_0), T_1 = \pi^{-1}(S_1) \cap T_0, V_0 = \psi^{-1}(T_1)$ and, finally, $U_0 \subset \mathbf{SL}_n$ be any $\mathbb{Q}$-defined Zariski open subset of $\phi(V_0)$. We are going to show that these U_0 and V_0 are as described in Proposition 5.

Indeed, let $K/\mathbb{Q}$ be a certain extension and $z \in (U_0)_K$. Obviously, ϕ coincides with the composite map $\rho \circ \pi \circ \psi$ where $\rho\colon S \to \mathbf{SL}_n$ is the projection to the second component. So, if we put $P = \rho^{-1}(z) \cap S_1$, $B = \pi^{-1}(P) \cap T_1$ then $\phi^{-1}(z) \cap V_0 = \psi^{-1}(B)$. However, P is a K-defined open subset of the affine space, and $\pi^{-1}(P)$ is a subvariety of the product $\mathbb{A}^n \times P$ defined by a linear system which satisfies the assumptions of Lemma 3 (the latter fact easily follows from our construction). So, by this lemma $\pi^{-1}(P)$ is an irreducible K-rational variety, and consequently, B is also. Now, for fixed $(y, z) \in B$ the fibre $\psi^{-1}(y, z)$ consists of $(x, y) \in \mathbf{GL}_n \times \mathbf{GL}_n$ such that

$$xy = (zy)x. \tag{8}$$

By our construction, y is regular semisimple and the characteristic polynomials of y and zy coincide, so, y and zy are conjugate in $\mathbf{GL}_n$. Moreover, the space of solutions of (8) in $\mathbf{M}_n$ has dimension n, i.e. the rank of the homogeneous linear system (8) is $n^2 - n$ for any point $(y, z) \in B$. So, again $\psi^{-1}(B)$ is an open subset in a subvariety of the product $\mathbf{M}_n \times B$ defined by a linear system satisfying Lemma 3, and therefore $\psi^{-1}(B)$ is irreducible and K-rational when B is. It also follows from Lemma 3 that $\dim B = \dim P + (n - (n-1)) = n^2 - n + 1$ and $\dim \psi^{-1}(B) = \dim B - (n^2 - (n^2 - n)) = n^2 + 1$. Proposition 5 is proved. ∎

Now, to complete the proof of Proposition 4 it remains to show that there is a smaller open set $U \subset U_0$ such that for $z \in U$ the whole fibre $\phi^{-1}(z)$ is irreducible. Assume the contrary. Then

$$C = \{z \in \mathbf{SL}_n \mid \phi^{-1}(z) \text{ is reducible}\}$$

is dense. Put $D = (\mathbf{GL}_n \times \mathbf{GL}_n) \backslash V_0$. If $\overline{\phi(D)} \neq \mathbf{SL}_n$ then for $z \in U_0 \cap (\mathbf{SL}_n \backslash \overline{\phi(D)})$ we would have $\phi^{-1}(z) \subset V_0$ and therefore $\phi^{-1}(z) = \phi^{-1}(z) \cap V_0$ would be irreducible by Proposition 5 which would contradict the density of C. So, we may (and we will) assume in addition that $\overline{\phi(D)} = \mathbf{SL}_n$.

Let $D = \bigcup_{i=1}^{d} D_i$ be a decomposition into irreducible components. Let $I = \{i \mid \overline{\phi(D_i)} = \mathbf{SL}_n\}$, $J = \{1, \ldots, d\} \setminus I$. For each $i \in I$ we consider

$$\phi_i = \phi \big|_{D_i} : D_i \to \mathbf{SL}_n,$$

and we let $P_i \subset \phi_i(D_i)$ be an open (in $\mathbf{SL}_n$) subset such that for any $z \in P_i$ we have

$$\dim \phi_i^{-1}(z) = \dim D_i - \dim \mathbf{SL}_n. \tag{9}$$

Put $W = (\bigcap_{i \in I} P_i) \bigcap (\mathbf{SL}_n \backslash (\bigcup_{j \in J} \overline{\phi(D_j)})) \cap U_0$ and pick $z \in W \cap C$. Since $\phi^{-1}(z) \cap V_0$ is irreducible but $\phi^{-1}(z)$ is not, there must be an irreducible component $E \subset \phi^{-1}(z)$ which does not meet V_0, i.e. lies in D. Besides, by our construction E in fact belongs to $\bigcup_{i \in I} D_i$, so $E \subset D_i$ for some $i \in I$. Then

$$\dim \phi_i^{-1}(z) \geq \dim E \geq n^2 + 1 = (\dim(\mathbf{GL}_n \times \mathbf{GL}_n) - \dim \mathbf{SL}_n)$$

implying by virtue of (10) that

$$\dim D_i = \dim \phi_i^{-1}(z) + \dim \mathbf{SL}_n \geq (n^2 + 1) + (n^2 - 1) = 2n^2 = \dim(\mathbf{GL}_n \times \mathbf{GL}_n)$$

which is impossible. The proof of Proposition 4 is complete. ∎

Remark: It would be interesting to show that the variety T introduced in the proof of Proposition 5 is irreducible. This would enable us to prove that for any z in some Zariski open subset of $\mathbf{SL}_n$ the variety $T(z) = \{y \in \mathbf{GL}_n \mid \sigma_i(y) = \sigma_i(zy), i = 1, \ldots, n-1\}$ is irreducible, this fact being slightly sharper than our assertion on the irreducibility of the generic commutator variety $W(z)$. In fact, our argument shows that one of the irreducible components of T is certainly $\overline{\mathrm{Im}\ \psi}$, and if there is any other component it cannot contain a point (y, z) with regular semisimple y. On the other hand, there do exist cases when both $T(z)$ and $W(z)$ are reducible.

Example: Let

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the equations for $y = (y_{ij})$ to belong to $T(z)$ are

$$y_{31} = 0, \quad y_{21}y_{32} = 0,$$

implying that $T(z)$ has two irreducible components. Obviously, both components contain regular semisimple elements, and therefore can be lifted to irreducible components of $W(z)$.

Along with the full commutator variety $W(z)$ ($z \in \mathbf{SL}_n$) one may consider its subvarieties:

$$W_1(z) = W(z) \cap (\mathbf{GL}_n \times \mathbf{SL}_n), \quad W_2(z) = W(z) \cap (\mathbf{SL}_n \times \mathbf{SL}_n).$$

Obviously, $W_2(z)$ arises in a natural way when one is considering representations of Γ into $\mathbf{SL}_n$ (cf. §4). To close this section, let us prove the following analog of Proposition 5 for $W_i(z)$ ($i = 1, 2$).

PROPOSITION 6: *There exists a $\mathbf{Q}$-defined Zariski open set $U' \subset \mathbf{SL}_n$ such that for any $z \in U'$ both varieties $W_1(z)$ and $W_2(z)$ are irreducible. Besides, for any extension $K/\mathbf{Q}$ and any $z \in U'_K$ the variety $W_1(z)$ is rational over K.*

Proof: Take for U' the intersection of U as constructed in Proposition 4 with the set of regular semisimple elements, and let $z \in U'$. Since $W(z)$ is irreducible and $W_1(z)$ is defined in $W(z)$ by a single equation (the determinant of the second component of a point $(x, y) \in W_1(z)$ should be one), any irreducible component of $W_1(z)$ has dimension $d = \dim W(z) - 1 = n^2$. Consider the morphism θ: $R \times W_1(z) \to W(z)$ where R is the one-dimensional torus of scalar matrices and $\theta(r, (x, y)) = (x, yr)$. Clearly, for any irreducible component $W' \subset W_1(z)$ we have $\overline{\theta(R, W')} = W(z)$ implying that $\theta(R', W') = W_1(z)$ for $R' = R \cap \mathbf{SL}_n$. So, it suffices to show that in fact $\theta(R', W') = W'$. Assume the contrary. Since z is regular semisimple, $W_1(z)$ contains a point (x, y) with x regular semisimple (such

x can already be found in the maximal torus containing z), and to begin with we may pick W' to be a component containing such (x, y). Moreover, the set of such points being open and consequently dense in W', we may assume in addition that (x, y) does not belong to any other component and $\theta(r, (x, y)) = (x, yr) \notin W'$ for some $r \in R'$. Consider the set $Z = (x, y(Z(x) \cap \mathbf{SL}_n))$. Since x is regular semisimple, this set is an irreducible subset of $W_1(z)$ containing a point which belongs to W' but not to any other irreducible component, and therefore $Z \subset W'$. However $r \in Z(x) \cap \mathbf{SL}_n$ and then $(x, yr) \in W'$ — a contradiction. To prove that $W_2(z)$ is irreducible we consider the imbedding $W_2(z) \subset W_1(z)$ and apply the same argument (note that again $W_2(z)$ is defined in $W_1(z)$ by a single equation).

The proof of K-rationality of $W_1(z)$ for $z \in U'_K$ is entirely analogous to the corresponding argument for $W(z)$. Namely, we introduce a subvariety $T' \subset \mathbf{GL}_n \times \mathbf{SL}_n$ of points (y, z) satisfying the system:

$$\left\{\begin{array}{lcl} \sigma_t(zy) & = & \sigma_t(y)\ ,\ t = 1, \ldots, n-1 \\ \det\ (y) & = & 1 \end{array}\right. \tag{10}$$

and its projection $\pi'\colon T' \to S$, $\pi' = \pi\ \big|_{T'}$ (π, S are as above). Arguing as in the proof of Proposition 5, one can show that (10) is equivalent to a system of n linear equations for the entries of the first column of y:

$$\sum_{s=1}^{n} c'_{st} y_{s1} = d'_t, \quad t = 1, \ldots, n \tag{11}$$

for some $c'_{st}, d'_t \in \mathbb{Q}[S]$. In fact, the first $(n-1)$ equations in (11) are the same as in (4), and the last one is the expansion of det y along the first column. Besides, using the same point (a, z) as in the proof of Proposition 5 one shows that a $\mathbb{Q}$-defined Zariski open set $T'_0 \subset T'$ of points (y, z) satisfying

(i)$'$ y is regular semisimple,

(ii)$'$ the rank of the matrix $C'(\pi'(y, z)) = (c_{ij}(\pi'(y, z)))_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ is n,

is non-empty (this easily follows from (6) and (7)). The rest of the argument repeats verbatim the concluding part of the proof of Proposition 5. ∎

The question of whether $W_2(z)$ is K-rational for $z \in U'_K$ is more delicate, and there are at least two obstructions to push through the above argument in this case. Namely, $W_2(z)$ can be viewed as a fibered space $\theta\colon W_2(z) \to B$, $\theta((x, y)) = y$, over

$$B = \{y \in \mathbf{SL}_n \mid \sigma_t(zy) = \sigma_t(y), \quad t = 1, \ldots, n-1\}.$$

We have shown above that B is K-rational, however if b is a generic point of B over K, then the fibre $\theta^{-1}(b)$ contains no rational points over $K(b)$ even for $n = 2$. Besides, $\theta^{-1}(b)$ is a principal homogeneous space of the "generic" norm torus $R^{(1)}_{L/K(b)}(G_m)$ which is not rational for $n \geq 4$ (cf. [C], [V], [Co-Sa]). This means that a birational parameterization of $W_2(z)$ (if there is any) must essentially be a simultaneous parameterization of both components x and y. This general observation is precisely consistent with what we do in the proof of the following:

LEMMA 4: *Let $z = \mathrm{diag}(\alpha, \alpha^{-1})$, $\alpha \in K^*$, $\alpha \neq -1$. Then*

$$W_2(z) = \{(x, y) \in \mathbf{SL}_2 \times \mathbf{SL}_2 | \, [x, y] = z\}$$

is K-rational.

Proof: The proof is purely computational. By virtue of Proposition 3 we may (and we will) assume that $\alpha \neq 1$. The variety B of y-components $y = (y_{ij})$ of elements from $W_2(z)$ is defined by the system:

$$\begin{cases} \mathrm{tr}(zy) & = & \mathrm{tr}(y), \\ \det(y) & = & 1, \end{cases} \quad \text{i.e.} \quad \begin{cases} (1-\alpha)y_{11} + (1-\alpha^{-1})y_{22} & = & 0, \\ y_{11}y_{22} - y_{12}y_{21} & = & 1. \end{cases}$$

Then the functions

$$y_{22} = \alpha y_{11}, \quad y_{21} = \frac{(\alpha y_{11}^2 - 1)}{y_{12}} \tag{12}$$

provide a birational parameterization of B. Furthermore, x-components are defined from the system

$$\begin{cases} xy & = & (zy)x, \\ \det\, x & = & 1. \end{cases} \tag{13}$$

Computations show that for fixed $y = (y_{ij})$ all $x = (x_{ij})$ satisfying the first equation in (13) admit the following parameterization (under the assumption that $y_{12} \neq 0$) :

$$x_{21} = \frac{(1-\alpha)y_{11}x_{11} + y_{21}x_{12}}{\alpha y_{12}}, \quad x_{22} = \frac{y_{12}x_{11} + (y_{22} - \alpha y_{11})x_{12}}{\alpha y_{12}}.$$

Taking into account (12), we now can rewrite the second equation of (13) in the form:

$$y_{12}^2 x_{11}^2 - (1-\alpha)y_{11}y_{12}x_{11}x_{12} - (\alpha y_{11}^2 - 1)x_{12}^2 - \alpha y_{12}^2 = 0. \tag{14}$$

Making a substitution $u = y_{12}x_{11}, v = y_{11}x_{12}$, we reduce (14) to

$$u^2 - (1-\alpha)uv - \alpha v^2 + x_{12}^2 - \alpha y_{12}^2 = 0. \tag{15}$$

Letting now $s = u - v, t = u + \alpha v$ (note that since $\alpha \neq -1$ this substitution is non-degenerate), we can rewrite (15) as

$$st + x_{12}^2 - \alpha y_{12}^2 = 0$$

which evidently defines a rational variety. Substituting back we obtain a parameterization of $W_2(z)$. The lemma is proved. ∎

The case $\alpha = -1$ is, indeed, exceptional. It is easy to show (cf. [T]) that $z = \operatorname{diag}(-1,-1)$ is not a commutator in $\mathbf{SL}_2(\mathbb{R})$. This means that $W_2(z)_{\mathbb{R}} = \emptyset$, in particular $W_2(z)$ cannot be $\mathbb{R}$-rational. However, if K is algebraically closed $W_2(z)$ is K-rational even for $z = \operatorname{diag}(-1,-1)$.

If K is not algebraically closed we, unfortunately, do not know at the moment whether $W_2(z)$ is K-rational for "generic" $z \in \mathbf{SL}_2(K)$. What we do know is that for "generic" $z \in \mathbf{SL}_n(K), n$ arbitrary, $W_2(z)$ is K-unirational (cf. §4). This result allows us, in particular, to give a very simple proof of the fact that if $K = \mathbb{C}$ (or any algebraically closed field) then for any regular semisimple $z \in \mathbf{SL}_2(K)$ the variety $W_2(z)$ is K-rational (which, of course, also follows from Lemma 4). Indeed, we may assume that z is diagonal, and then the torus $S = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\}$ acts on $W_2(z)$ by conjugation, the isotropy subgroup of a "generic" point being trivial. Consider the quotient $X = W_2(z)/S$. Then X is a K-unirational surface, and so X is, in fact, rational since K is algebraically closed [Sh2]. Now using the fact that S has trivial cohomology over any field, one easily derives from this K-rationality of $W_2(z)$.

3. Proof of Theorem 1

Throughout this section let $\Gamma = \Gamma_g$ be the group with $2g$ generators $x_1, y_1, \ldots, x_g, y_g$ and a single defining relation

$$[x_1, y_1] \cdots [x_g, y_g] = 1$$

for some $g > 1$, $\mathbf{R}_n = \mathbf{R}_n(\Gamma)$ be the corresponding variety of n-dimensional representations. Put $P = (\mathbf{GL}_n)^{2g-2}$ and let $\phi\colon \mathbf{R}_n \to P$ be the projection onto the first $(2g-2)$ components. (Note the following "natural" interpretation of ϕ in terms of representation varieties: elements $x_1, y_1, \ldots, x_{g-1}, y_{g-1}$ freely generate a subgroup $F \subset \Gamma$ (cf. [L-S], Ch. 2), so P can be identified with $\mathbf{R}_n(F)$, and with this identification ϕ becomes nothing but the restriction map $\mathbf{R}_n(\Gamma) \to \mathbf{R}_n(F)$.)

Since any element in $\mathbf{SL}_n$ is a commutator [T], one can see easily that ϕ is surjective and, in particular, dominant. The proof of the irreducibility of $\mathbf{R}_n$ rests on the fact that ϕ remains dominant if restricted to an arbitrary irreducible component of $\mathbf{R}_n$.

PROPOSITION 7: *For any irreducible component $V \subset \mathbf{R}_n$ we have $\overline{\phi(V)} = P$.*

We assume for a moment Proposition 7, and show how this implies the irreducibility of $\mathbf{R}_n$. Let $\mathbf{R}_n = \bigcup_{i=1}^{d} V_i$ be the decomposition into the union of irreducible components, and let $d > 1$. Define κ: $P \to \mathbf{SL}_n$ by the formula

$$\kappa\left((x_1, y_1, \ldots, x_{g-1}, y_{g-1})\right) = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}].$$

Also, let ψ, as in the previous section, be the commutator map $\mathbf{GL}_n \times \mathbf{GL}_n \to \mathbf{SL}_n$ sending (x, y) to $[x, y]$, and $U \subset \mathbf{SL}_n$ be a Zariski open set such that the fibre $\psi^{-1}(z)$ is irreducible for any $z \in U$ (cf. Proposition 4). Put $U_i = V_i \setminus (\bigcup_{j \neq i} V_j)$ $(i, j = 1, \ldots, d)$ and $U_0 = \kappa^{-1}(U)$. Since P is irreducible, the intersection $\phi(U_1) \cap \phi(U_2) \cap U_0$ is non-empty; let a be any of its points. Then the fibre $Z = \phi^{-1}(a)$ is isomorphic to the commutator variety $W(\kappa(a))$, and therefore is irreducible. Hence $Z \subset V_{i_0}$ for a suitable $i_0 \in \{1, \ldots, d\}$. But $a = \phi(u_1) = \phi(u_2)$ for some $u_i \in U_i$ $(i = 1, 2)$, so $u_1, u_2 \in Z$. However each of u_1, u_2 lies on a single irreducible component of $\mathbf{R}_n$ implying $V_1 = V_{i_0} = V_2$ — a contradiction.

The proof of Proposition 7 is based on the analysis of the differential $d_v\phi$.

LEMMA 5: *Let $v = (x_1, y_1, \ldots, x_g, y_g) \in \mathbf{R}_n$ be a point such that x_g and y_g are regular elements and $\dim(Z(x_g) \cap Z(y_g)) = 1$ (i.e. $Z(x_g) \cap Z(y_g)$ consists of scalar matrices only). Then v is a simple point on $\mathbf{R}_n$ and the map $d_v\phi$: $T_v(\mathbf{R}_n) \to T_{\phi(v)}(P)$ is surjective.*

Proof: For $x, y \in \mathbf{GL}_n$ let $\tau_{x,y}$ denote the map $\mathbf{M}_n \times \mathbf{M}_n \to \mathbf{M}_n^0 = \{X \in \mathbf{M}_n | \operatorname{tr}(X) = 0\}$ defined by the formula

$$\tau_{x,y}(X, Y) = (y^{-1}Xy - X) + (Y - x^{-1}Yx).$$

LEMMA 6: *If $x, y \in \mathbf{GL}_n$ are regular matrices such that $Z(x) \cap Z(y)$ consists of scalar matrices only, then $\tau_{x,y}$ is surjective.*

Proof: Put

$$V_1 = \{x^{-1}Xx - X | X \in \mathbf{M}_n\}, \quad V_2 = \{y^{-1}Yy - Y | Y \in \mathbf{M}_n\},$$

and let $f(X, Y)$ denote the non-degenerate bilinear form on $\mathbf{M}_n$ given by

$$f(X, Y) = \mathrm{tr}(XY).$$

We have to prove that $V_1 + V_2 = \mathbf{M}_n^0$. Assume the contrary. Since both x and y are regular, we have $\dim V_1 = \dim V_2 = n^2 - n$. Furthermore,

$$\dim \mathbf{M}_n^0 = n^2 - 1 > \dim(V_1 + V_2) = 2(n^2 - n) - \dim(V_1 \cap V_2)$$

implying

$$\dim(V_1 \cap V_2) > n^2 - (2n - 1). \tag{16}$$

Now, for $A \in \mathbf{M}_n$ let f_A denote the linear functional on $\mathbf{M}_n$ defined by

$$f_A(X) = f(X, A).$$

Then each of the functionals $f_{x^0}, f_x, \dots, f_{x^{n-1}}$ vanishes on V_1, so does each of $f_{y^0}, f_y, \dots, f_{y^{n-1}}$ on V_2. This implies that all $(2n - 1)$ functionals

$$f_{x^0}, f_x, \dots, f_{x^{n-1}}, f_y, \dots, f_{y^{n-1}} \tag{17}$$

vanish on $V_1 \cap V_2$. Now it follows from (16) that the functionals in (17) cannot be linearly independent, and therefore, since f is non-degenerate, we conclude that the elements

$$x^0, x, \dots, x^{n-1}, y, \dots, y^{n-1}$$

must be linearly dependent. In other words, there must be a relation of the form

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} = \beta_1 y + \cdots + \beta_{n-1} y^{n-1} \tag{18}$$

in which not all of $\alpha_0, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}$ are equal to zero. Let a be the element given by either side of (18). Clearly, a commutes with both x and y, and to derive a contradiction, it remains to notice that a is not a scalar matrix, which easily follows from the fact that neither x nor y can satisfy a non-trivial polynomial equation of degree less than n.

Lemma 6 is proved. ∎

Let us proceed with the proof of Lemma 5. A direct computation with dual numbers shows that the differential $d_{(x,y)}\psi$ of the commutator map ψ: $\mathbf{GL}_n \times \mathbf{GL}_n \to \mathbf{SL}_n$, $(x, y) \mapsto [x, y]$, at point (x, y) is given by the formula

$$\begin{aligned} (d_{(x,y)}\psi)(X, Y) &= xy\tau_{x,y}(x^{-1}X, y^{-1}Y)x^{-1}y^{-1} \\ &= [x, y](yx)\tau_{x,y}(x^{-1}X, y^{-1}Y)(yx)^{-1} \end{aligned} \tag{19}$$

(note that the tangent space $T_{\psi(x,y)}(\mathbf{SL}_n)$ to $\mathbf{SL}_n$ at $\psi(x,y) = [x,y]$ can be identified with $[x,y]\mathbf{M}_n^0$). Using (19), it is easy to produce a formula for the differential of the multi-commutator map μ: $(\mathbf{GL}_n)^{2g} \to \mathbf{SL}_n$

$$\mu((x_1, y_1, \ldots, x_g, y_g)) = [x_1, y_1] \cdots [x_g, y_g],$$

at an arbitrary point $v = (x_1, y_1, \ldots, x_g, y_g)$:

$$d_v\mu(X_1, Y_1, \ldots, X_g, Y_g) = \sum_{i=1}^{g}([x_1, y_1] \cdots [x_i, y_i]\ Z_i\ [x_{i+1}, y_{i+1}] \cdots [x_g, y_g]),$$

where $Z_i = (y_i x_i)\tau_{x_i, y_i}(x_i^{-1}X_i, y_i^{-1}Y_i)(y_i x_i)^{-1}$ (of course, for $i = g$ the product $[x_{i+1}, y_{i+1}] \cdots [x_g, y_g]$ should be omitted). Since $\mathbf{R}_n = \mu^{-1}(e)$, for any $v \in \mathbf{R}_n$ on $T_v(\mathbf{R}_n)$ we identically have

$$d_v\mu(X_1, Y_1, \ldots, X_g, Y_g) = 0. \tag{20}$$

It should be noted that though $\mathbf{R}_n$ is defined by a single matrix equation $\mu((x_1, y_1, \ldots, x_g, y_g)) = E_n$, we cannot say in general that $T_v(\mathbf{R}_n)$ is defined by (20) (cf. [Lu-M] for an example when the tangent space to the representation variety of a group is not defined by the differentials of relations defining the group). Nevertheless (20) does define $T_v(\mathbf{R}_n)$ in one special case we are going to describe. Since μ is surjective, it follows from the Dimension Theorem that the dimension of any irreducible component of $\mathbf{R}_n$ is $\geq 2gn^2 - (n^2 - 1) = (2g - 1)n^2 + 1$; in particular, for any $v \in \mathbf{R}_n$ we have $\dim T_v(\mathbf{R}_n) \geq (2g - 1)n^2 + 1$. Therefore, if the space defined by (20) has dimension $(2g - 1)n^2 + 1$, it must coincide with $T_v(\mathbf{R}_n)$ and the point v is simple.

Now, let us rewrite (20) in the form

$$\begin{aligned} &\tau_{x_g, y_g}(x_g^{-1}X_g\ ,\ y_g^{-1}Y_g) \\ &= -(y_g x_g)^{-1}\left(\textstyle\sum_{i=1}^{g-1} ([x_g, y_g]^{-1} \cdots [x_{i+1}, y_{i+1}]^{-1} Z_i \right. \\ &\qquad \left. \times [x_{i+1}, y_{i+1}] \ldots [x_g, y_g])\right)(y_g x_g)^{-1}, \end{aligned} \tag{21}$$

and let $\theta(X_1, Y_1, \ldots, X_{g-1}, Y_{g-1})$ denote the right-hand side of (21). Clearly, θ takes on values in $\mathbf{M}_n^0$. Suppose x_g and y_g satisfy the assumptions made in the statement of Lemma 5. Then by Lemma 6, τ_{x_g, y_g}: $\mathbf{M}_n \times \mathbf{M}_n \to \mathbf{M}_n^0$ is surjective, implying that for any tuple $(X_1, Y_1, \ldots, X_{g-1}, Y_{g-1}) \in (\mathbf{M}_n)^{2g-2}$ one can find a pair $(X_g, Y_g) \in \mathbf{M}_n \times \mathbf{M}_n$ which satisfies (21). This means that if S is the space of solutions of (20), then $d_v\phi$ (equal to the projection to the first $2g-2$ components) maps S surjectively onto $(\mathbf{M}_n)^{2g-2} = T_{\phi(v)}(P)$, and it remains to be shown that v is a simple point and $S = T_v(\mathbf{R}_n)$. To do this, it suffices to prove that $\dim S = (2g-1)n^2+1$.

We have

$$\begin{aligned}\dim S =& \dim(\mathbf{M}_n)^{2g-2} + \dim \operatorname{Ker} \tau_{x_g, y_g} \\ =&(2g-2)n^2 + (2n^2 - (n^2-1)) = (2g-1)n^2+1\end{aligned}$$

as required. Lemma 5 is proved. ∎

Lemma 7: *Let V be an irreducible component of $\mathbf{R}_n$. If $\overline{\phi(V)} \neq P$, then*

$$\dim(Z(x_g) \cap Z(y_g)) > 1 \quad \textit{for any } v = (x_1, y_1, \ldots, x_g, y_g) \in V.$$

Proof: Put

$$\begin{aligned} U_1 &= \{v = (x_1, y_1, \ldots, x_g, y_g) \in V \mid x_g \text{ and } y_g \text{ are regular}\}, \\ U_2 &= \{v = (x_1, y_1, \ldots, x_g, y_g) \in V \mid \dim Z(x_g) \cap Z(y_g) = 1\}. \end{aligned}$$

It is easy to see that both U_1 and U_2 are Zariski open. To begin with, let us show that $U_1 \neq \emptyset$. Let $v^0 = (x_1^0, y_1^0, \ldots, x_g^0, y_g^0) \in V$ be a point which does not belong to any other irreducible component of $\mathbf{R}_n$, and let W^0 be an irreducible component of the commutator variety $W(z^{-1})$, $z = [x_1^0, y_1^0] \cdots [x_{g-1}^0, y_{g-1}^0]$, containing (x_g^0, y_g^0). Then $(x_1^0, y_1^0, \ldots, x_{g-1}^0, y_{g-1}^0, W^0) \subset V$; on the other hand, W^0 contains a point (x_g, y_g) with regular x_g and y_g (cf. Corollary in §1), hence our claim. If we assume that $U_2 \neq \emptyset$ then $V_0 = U_1 \cap U_2 \neq \emptyset$. It follows from Lemma 5 that any point $v \in V_0$ is simple on $\mathbf{R}_n$ implying, in particular, that $T_v(V) = T_v(\mathbf{R}_n)$, and $d_v\phi$: $T_v(\mathbf{R}_n) = T_v(V) \to T_{\phi(v)}(P)$ is surjective. There exists $v \in V_0$ such that $\phi(v)$ is a simple point on $Z = \overline{\phi(V)}$, and then $\dim Z = \dim T_{\phi(v)}(Z) \geq \dim(d_v\phi)(T_v(\mathbf{R}_n)) = \dim T_{\phi(v)}(P) = \dim P$, so $Z = P$, contradicting the assumptions of the lemma.

Therefore $U_2 = \emptyset$, and Lemma 7 is proved. ∎

Now, we are going to show that if for an irreducible component $V \subset \mathbf{R}_n$ we have $\overline{\phi(V)} \neq P$, then the condition on points of V provided by Lemma 7 forces the dimension of V to be less than an obvious lower bound, thereby completing the proof of irreducibility of $\mathbf{R}_n$ (of course, in doing this we may (and we will) assume that $n > 1$).

LEMMA 8:

(i) $\dim V \geq (2g-1)n^2+1$;

(ii) *for any* $z \in \mathbf{SL}_n$ *the dimension of any irreducible component* T *of the commutator variety* $W(z)$ *is between* (n^2+1) *and* (n^2+n);

(iii) *the dimension of the variety*

$$Z = \{(x,y) \in \mathbf{GL}_n \times \mathbf{GL}_n \mid \dim(Z(x) \cap Z(y)) > 1\}$$

is $\leq 2n^2 - 2(n-1)$.

Proof: (i) has already been proved in the course of the proof of Lemma 5. The lower bound in (ii) follows from the Dimension Theorem applied to the commutator map ψ. To prove the upper bound, consider the map $\delta\colon T \to \mathbf{GL}_n$ given by the projection $(x,y) \to x$. It follows from the Corollary in §1 that $\delta(T)$ contains a regular element x_0. Then the fibre $\delta^{-1}(x_0)$ has dimension n, and therefore

$$\dim T \leq \dim \overline{\delta(T)} + \dim \delta^{-1}(x_0) \leq n^2+n.$$

(iii) We show first that there are finitely many non-scalar matrices $x_1, \ldots, x_r \in \mathbf{GL}_n$ with the following property: for an arbitrary non-scalar $x \in \mathbf{GL}_n$ a suitable conjugate of $Z(x)$ is contained in one of $Z(x_i), i = 1, \ldots, r$. To do so, let us fix two elements $\alpha \neq \beta$ in the ground field and for each i, $1 \leq i \leq n-1$, introduce an element

$$x = \mathrm{diag}(\underbrace{\alpha, \ldots, \alpha}_{i}, \underbrace{\beta, \ldots, \beta}_{n-i}).$$

Furthermore, let $x_n, \ldots, x_r$ be representatives of all non-identity unipotent conjugacy classes in $\mathbf{GL}_n$ (it is well-known that there are finitely many such classes in an arbitrary reductive group in characteristic zero [Sp-St]; for $\mathbf{GL}_n$ this easily follows from the existence of the Jordan normal form). Now, let $x \in \mathbf{GL}_n$ be an arbitrary non-scalar matrix, and $x = x_s x_u$ be its Jordan decomposition; then $Z(x) \subset Z(x_s) \cap Z(x_u)$. If $x_u \neq E_n$ then x_u is conjugate to one of x_i $(n \leq i \leq r)$. Otherwise, $x = x_s$ is conjugate to a non-scalar diagonal matrix d of the form

$$d = \mathrm{diag}(\underbrace{\alpha_1, \ldots, \alpha_1}_{n_1}, \ldots, \underbrace{\alpha_l, \ldots, \alpha_l}_{n_l})$$

where all α_j are distinct, and then $Z(d) \subset Z(x_{n_1})$. Now, we can write Z in the form

$$Z = \bigcup_{i=1}^{r} \bigcup_{h \in \mathbf{GL}_n} h(Z(x_i) \times Z(x_i))h^{-1}.$$

So, it suffices to show that for any $x = x_i$ the image of the morphism

$$\eta\colon S = \mathbf{GL}_n \times Z(x) \times Z(x) \to \mathbf{GL}_n \times \mathbf{GL}_n, \ \eta\colon (h, z_1, z_2) \mapsto (hz_1h^{-1}, hz_2h^{-1}),$$

has dimension $\leq 2n^2 - 2(n-1)$. However, for fixed $z_1, z_2 \in Z(x)$, $h \in \mathbf{GL}_n$ and an arbitrary $t \in Z(x)$ we have

$$\eta(ht, t^{-1}z_1\, t,\ t^{-1}z_2\, t) = \eta(h,\ z_1,\ z_2)$$

implying that for any $s \in S$ we have $\dim \eta^{-1}(\eta(s)) \geq \dim Z(x)$ and therefore

$$\dim \operatorname{Im} \eta \leq \dim S - \dim Z(x) = n^2 + \dim Z(x).$$

So, it remains to be shown that $\dim Z(x) \leq n^2 - 2(n-1)$. If $x = x_i, 1 \leq i \leq n-1$, then $Z(x) = \mathbf{GL}_i \times \mathbf{GL}_{n-i}$ and

$$\begin{aligned}\dim Z(x) &= i^2 + (n-i)^2 = n^2 - 2i(n-i)\\ &= n^2 - 2(n-1) + 2(i-1)(i-(n-1)) \leq n^2 - 2(n-1).\end{aligned}$$

If $x = x_i, n \leq i \leq r$, then x is unipotent. Pick an integer $l > 0$ such that $h = (x - E_n)^l \neq 0$ but $(x - E_n)^{l+1} = 0$. Then $Z(x) \subset Z(h)$ and $h^2 = 0$, so it suffices to estimate $\dim Z(x)$. Clearly, h is conjugate to a Jordan matrix of the form

$$a = \operatorname{diag}(\underbrace{j, \ldots, j}_{r},\ O_{n-2r})$$

for some $r > 0$, where $j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

A simple computation shows that $\dim Z(a) = n^2 - 2r(n-r)$ yielding the required estimation as above. Lemma 8 is proved. ∎

It follows from Lemma 7 that if $\overline{\phi(V)} \neq P$, then $V \subset (\mathbf{GL}_n)^{2g-2} \times Z$ where Z is the variety introduced in Lemma 8 (iii). Consider the map

$$\begin{aligned}&\mu\colon V \to (\mathbf{GL}_n)^{2g-2},\\ &\mu\colon (x_1, y_1, \ldots, x_{g-1}, y_{g-1}, x_g, y_g) \ \mapsto (x_1, y_1, \ldots, x_{g-2}, y_{g-2}, x_g, y_g).\end{aligned}$$

Clearly, $\operatorname{Im}\mu \subset (\mathbf{GL}_n)^{2g-4} \times Z$ and therefore by Lemma 8 (iii) we have

$$\dim \operatorname{Im}\mu \leq (2g-4)n^2 + 2n^2 - 2(n-1) = (2g-2)n^2 - 2(n-1).$$

On the other hand, the fibres of μ are the commutator varieties, so by Lemma 8(ii) $\dim \mu^{-1}(\mu(v)) \leq n^2 + n$ for any $v \in V$ and finally

$$\dim V \leq (2g-2)n^2 - 2(n-1) + n^2 + n = (2g-1)n^2 - n + 2. \tag{22}$$

Comparing (22) with Lemma 8(i) we get $n \leq 1$ — a contradiction, proving irreducibility of $\mathbf{R}_n$.

The assertion about $\dim \mathbf{R}_n$ has already been (implicitly) proved above. Indeed, by Lemma 8 (i) $\dim \mathbf{R}_n \geq (2g-1)n^2 + 1$; on the other hand, in proving Lemma 5 we saw that there are points $v \in \mathbf{R}_n$ such that $\dim T_v(\mathbf{R}_n) = (2g-1)n^2 + 1$ giving the equality $\dim \mathbf{R}_n = (2g-1)n^2 + 1$. Of course, one can give another (direct and easy) proof of this fact just by noticing that there exists a Zariski open subset $P_0 \subset P$ such that $\dim \phi^{-1}(p) = n^2 + 1$ for any $p \in P_0$ (for such a P_0 one can take the U_0 introduced right after the statement of Proposition 7).

Now it is easy to complete the proof of Theorem 1. Let us consider "generic" $(n \times n)$-matrices $x_1, y_1, \ldots, x_{g-1}, y_{g-1}$ and denote as K the field generated over $\mathbb{Q}$ by their entries. Let

$$h = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \in \mathbf{SL}_n(K).$$

Then h^{-1} is a generic point of $\mathbf{SL}_n$ over $\mathbb{Q}$ and therefore h belongs to the $\mathbb{Q}$-defined Zariski open set $U \subset \mathbf{SL}_n$ specified in Proposition 5. By that proposition the commutator variety $W(h^{-1})$ is absolutely irreducible and rational over K. Since obviously $\mathbb{Q}(\mathbf{R}_n) = K(W(h^{-1}))$, we conclude that $\mathbf{R}_n$ is rational over $\mathbb{Q}$.

Proof of Theorem 2: There is a dominant (in fact, surjective) $\mathbb{Q}$-defined map $\sigma\colon \mathbf{R}_n(\Gamma) \to \mathbf{X}_n(\Gamma)$ whose fibers are the closures of orbits of the natural action of $\mathbf{GL}_n$ on $\mathbf{R}_n(\Gamma)$. So, for the first assertion it remains only to verify the dimension formulas. If $g = 1$, the stabilizer in $\mathbf{GL}_n$ of the generic point of $\mathbf{R}_n(\Gamma)$ is a maximal torus implying that the generic orbit has dimension $n^2 - n$, hence our claim. Now, for $g > 1$ there always exists an irreducible representation $\rho \in \mathbf{R}_n(\Gamma)$. (Indeed, $\mathbf{SL}_2$ is known to have an irreducible representation in every dimension; on the other hand, Γ is isomorphic to a Zariski dense subgroup of

$\mathbf{SL}_2$.) Then obviously the dimension of the orbit of ρ under $\mathbf{GL}_n$ is n^2-1, yielding the desired formula.

For the second assertion, consider the following commutative diagram

$$\begin{array}{ccc} R_n(\Gamma) & \xrightarrow{\sigma} & X_n(\Gamma) \\ \phi\downarrow & & \downarrow\delta \\ R_n(F) & \xrightarrow{\tau} & X_n(F) \end{array} \tag{23}$$

in which ϕ, as before, is obtained by restricting representations to the subgroup $F \subset \Gamma$ (freely) generated by $x_1, y_1, \ldots, x_{g-1}, y_{g-1}$; δ is the morphism of the character varieties corresponding to ϕ, and τ is the canonical projection. Let ξ be the generic point of $\mathbf{X}_n(F)$ over $\mathbb{Q}$ (so that $K = \mathbb{Q}(\xi)$ coincides with $\mathbb{Q}(\mathbf{X}_n(F))$) and $Z = \delta^{-1}(\xi)$ be the generic fiber of δ. Then the field $\mathbb{Q}(\mathbf{X}_n(\Gamma))$ can be identified with $K(Z)$. Since $\mathbf{X}_n(F)$ is rational over $\mathbb{Q}$ for $n \leq 4$ (cf. [F1], [F2]) it remains to be proved that Z is K-rational.

Let $\mu \in \tau^{-1}(\xi)$ and $M = \phi^{-1}(\mu)$ (note that μ and M are defined over some extension of K). We claim that σ induces a bijection (and consequently, a $\bar{K}$-defined birational isomorphism) between M and Z. Indeed, since F has irreducible representations in every dimension, the representation corresponding to μ is such. Then for any $m \in M$ the corresponding representation of Γ is also irreducible, and from $\sigma(m_1) = \sigma(m_2)$ for $m_1, m_2 \in M$ we may conclude that m_1 and m_2 determine equivalent representations of Γ, i.e. $m_2 = (\operatorname{Int} g)m_1$ for some $g \in \mathbf{GL}_n$. However both m_1 and m_2 restrict to the same representation μ of F implying that g belongs to the centralizer of μ , so g is, in fact, a scalar matrix and $m_1 = m_2$. This fact implies that Z is birationally $\bar{K}$-isomorphic to a commutator variety in $\mathbf{GL}_n$ in the generic position and therefore is rational at least over $\bar{K}$. However what we really want is the rationality over K, and then to overcome the fact that μ is **not** K-defined we have to replace $\mathbf{GL}_n$ by its suitable Galois twist G over K.

Since ξ is defined over K, for any $\theta \in \mathrm{Gal}(\bar{K}/K)$ we have $\tau(\theta(\mu)) = \tau(\mu)$, hence $\theta(\mu) = a_\theta(\mu)$ for a suitable uniquely defined $a_\theta \in \mathbf{PSL}_n$, since μ is irreducible. The family $a = \{a_\theta\}$ determines a cocycle in $Z^1(K, \mathbf{PSL}_n)$ and we can introduce the corresponding twisted group $G = {}^a\mathbf{GL}_n$. Then we can think of the representation variety $\mathbf{R}(\Gamma, G)$ (resp., $\mathbf{R}(F, G)$) as the Galois twist of $\mathbf{R}_n(\Gamma)$

(resp., $\mathbf{R}_n(F)$) by a, and consider the diagram

$$\begin{array}{ccc} R(\Gamma, G) & \xrightarrow{\hat{\sigma}} & X(\Gamma, G) = X_n(\Gamma) \\ {\scriptstyle\hat{\phi}}\downarrow & & \downarrow{\scriptstyle\hat{\delta}=\delta} \\ R(F, G) & \xrightarrow{\hat{\tau}} & X(F, G) = X_n(F) \end{array}$$

which is obtained by twisting (23) (note that twisting by inner automorphisms we do not change the corresponding character varieties). Then μ corresponds to a K-defined point (to be denoted by the same letter) in $\hat{\tau}^{-1}(\xi)$, and the same argument as above shows that the fibre $\delta^{-1}(\xi) = \hat{\delta}^{-1}(\xi)$ is birationally K-isomorphic to the fibre $\hat{\phi}^{-1}(\mu)$. On the other hand, obviously, $\hat{\phi}^{-1}(\mu)$ equals the following commutator variety in G in the generic position:

$$\hat{W}(z) = \{(x, y) \in G \times G | \; [x, y] = z\}$$

where $z = ([x_1, y_1] \cdots [x_{g-1}, y_{g-1}])^{-1}$ if $\mu = (x_1, y_1, \ldots, x_{g-1}, y_{g-1}), x_i, y_i \in G$. It is well-known (cf., for example, [Pl-R]) that $G = \mathbf{GL}_1(D)$ for some simple central algebra D over K of dimension n^2, so it remains to be proven that for $n \leq 3$ there exists a Zariski K-open set $U \subset \mathbf{SL}_1(D)$ with the following property: for any extension L/K and any $z \in U_L$ the commutator variety $\hat{W}(z)$ is L-rational. To this end, let us introduce the functions $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ on D which are similar to the functions $\sigma_1, \ldots, \sigma_n$ on $\mathbf{M}_n$ used in §2, and for $z \in \mathbf{SL}_1(D)$ define the corresponding variety

$$\hat{T}(z) = \{y \in G | \; \hat{\sigma}_i(y) = \hat{\sigma}_i(zy), i = 1, \ldots, n-1\}.$$

It is well known that $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$ are regular functions on D, satisfying the following:

(i) $\hat{\sigma}_i(d)$ is given by a homogeneous K-defined polynomial of degree i in the coefficients of d with respect to a basis of D over K;

(ii) if C is an algebraically closed field containing K and $\chi\colon D \otimes_K C \to \mathbf{M}_n(C)$ is an isomorphism of C-algebras (which always exists, cf. [P]), then $\hat{\sigma}_i(d) = \sigma_i(\chi(d))$ for any $d \in D$.

We claim that if $n \leq 3$ then for "generic" z the variety $\hat{T}(z)$ is rational over the field of definition. Indeed, if $n = 2$ then $\hat{T}(z)$ is defined by a single linear condition, and our claim follows immediately. For $n = 3$ we have two conditions, one being linear and another quadratic in the coefficients of y. Solving the linear equation for one of the variables and plugging this expression into the quadratic equation, we see that $\hat{T}(z)$ is actually a quadric. Therefore, to establish its rationality it suffices to find on it a rational point (over L, if $z \in \mathbf{SL}_1(D)_L$).

Let M be a maximal subfield in $D \otimes_K L$, and C be an algebraically closed field containing M. Since M splits $D \otimes_K L$, an isomorphism χ in (ii) can be found over M. Then $\hat{T}(z)$ becomes M-isomorphic to

$$T(\chi(z)) = \{y \in \mathbf{GL}_n \mid \sigma_i(y) = \sigma_i(\chi(z)y)\},$$

and since by virtue of Thompson's theorem $T(\chi(z))_M \neq \emptyset$ (cf. §2) we may conclude that $\hat{T}(z)_M \neq \emptyset$. This means that the quadratic equation defining $\hat{T}(z)$ has a rational point over an extension M of L of degree 3, but then by Springer's theorem [La] it must have a point over L, proving our claim.

The rest of the proof coincides with the final part in the proof of Proposition 6. Namely, for fixed z we may consider the natural map ψ: $\hat{W}(z) \to \hat{T}(z)$, ψ: $(x,y) \mapsto y$. Then the fibre $\psi^{-1}(y)$ equals the variety of x's in G satisfying $xy = (zy)x$ which amounts to a system of linear equations for the coefficients of x with respect to a basis of D over K. In view of Lemma 3 this implies rationality of $\psi^{-1}(y)$ for z (resp., y) in a Zariski open subset of $\mathbf{SL}_1(D)$ (resp., $\hat{T}(z)$). Applying this to the fibre of ψ over the generic point of $\hat{T}(z)$ (note that as (implicitly) proved above, $\hat{T}(z)$ is irreducible), we obtain the rationality of $\hat{W}(z)$. Theorem 2 is proved. ∎

Remark: In [R-Be] we announced a stronger statement than our Theorem 2, viz., we claimed that $\mathbf{X}_n(\Gamma_g)$ is stably isomorphic to $\mathbf{X}_n(F)$. However, in our original proof (sketched in [R-Be]) we overlooked that the fibre $\delta^{-1}(\xi)$ is isomorphic to the corresponding fibre of ϕ only over some field extension. In fact, the examples (due to Kursov [K]) of division algebras D having an element in the derived subgroup $[D^*, D^*]$ which is not a single commutator suggest that in general $\delta^{-1}(\xi)$ may not even have rational points.

4. Representations in $\mathbf{SL}_n$

The goal of this section is to prove the following (partial) analog of Theorem 1 for the representation variety $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)$ of Γ_g into $\mathbf{SL}_n$.

THEOREM 3: $\mathbf{R}(\Gamma_g, \mathbf{SL}_n)$ *is an (absolutely) irreducible* $\mathbf{Q}$*-unirational variety of dimension*

$$\dim \mathbf{R}(\Gamma_g, \mathbf{SL}_n) = \begin{cases} (2g-1)(n^2-1) & \text{if } g > 1, \\ n^2+n-2 & \text{if } g = 1. \end{cases}$$

The case $g = 1$ is simple. Indeed, as we mentioned in §2, Richardson [Ri] established the irreducibility of $\mathbf{R}(\Gamma_g, G) = \mathbf{R}(\mathbf{Z}^2, G)$ for an arbitrary reductive G (in fact, for $G = \mathbf{SL}_n$ this can be easily deduced from the irreducibility of $\mathbf{R}_n(\mathbf{Z}^2)$ by imitating the argument used in the proof of Proposition 6). Applying the Dimension Theorem to the projection $\mathbf{R}(\mathbf{Z}^2, G) \to G$, $(x, y) \mapsto x$, we immediately find the dimension of $\mathbf{R}(\mathbf{Z}^2, G)$ to be $\dim G + \operatorname{rk} G$. Now, let $T \subset G$ be a maximal torus defined over K, the field of definition of G. Then the dimension argument shows that the morphism $\theta\colon G \times T \times T \to \mathbf{R}(\mathbf{Z}^2, G)$ defined by

$$\theta(g, t_1, t_2) = (gt_1g^{-1}, gt_2g^{-1})$$

is dominant. Therefore, since G and T are K-unirational [Bo], so is $\mathbf{R}(\mathbb{Z}^2, G)$.

In the rest of this section $g > 1$. One can check that all the arguments from the previous section can be carried over to our situation provided we have the following analog of Proposition 4.

PROPOSITION 8: *There exists a* $\mathbb{Q}$*-defined Zariski open subset* $U' \subset \mathbf{SL}_n$ *such that for any extension* $K/\mathbb{Q}$ *and any point* $z \in U'_K$ *the variety* $W'(z) = \{(x, y) \in \mathbf{SL}_n \times \mathbf{SL}_n \mid [x, y] = z\}$ *is (absolutely) irreducible and unirational over K.*

(Note that the irreducibility of the "generic" commutator variety in $\mathbf{SL}_n$ has already been proved in Proposition 6.)

To begin with, let us outline the main idea of the proof. It was proved by Thompson [T] (cf. also Proposition 9 below) that almost any $z \in \mathbf{SL}_n(K)$ (in fact, any z which does not belong to the centre) is a commutator of two matrices $x, y \in \mathbf{SL}_n(K)$ i.e. $W'(z)_K \neq \emptyset$. Our proof of Proposition 8 is based on the fact that "generically" (in the sense to be specified later) any other point of $W'(z)$ can be obtained from (x, y) by means of a natural procedure which can be interpreted as a motion along some unirational subvariety. For technical reasons, it is more convenient to define and analyze this procedure first for the elements in $\mathbf{GL}_n$.

We will call two pairs $a_i = (x_i, y_i) \in \mathbf{GL}_n \times \mathbf{GL}_n$ $(i = 1, 2)$ **equivalent** if $[x_1, y_1] = [x_2, y_2]$ and either $x_1 = x_2$ or $y_1 = y_2$. This means that a_2 is obtainable from a_1 by a single **standard transformation** by which we mean multiplication of either of the components by an element from the centralizer of the other. Now, let $R \subset \mathbf{GL}_n$ be the set of regular semi-simple elements. Then $a, b \in \mathbf{GL}_n \times \mathbf{GL}_n$ (resp., $a, b \in R \times R$) are said to be **chain equivalent** (resp., **strictly chain equivalent**) if there are elements $c_1, \ldots, c_k$ in $\mathbf{GL}_n \times \mathbf{GL}_n$ (resp., in $R \times R$) such that $c_1 = a$, $c_k = b$ and for any $i = 1, \ldots, k-1$ the elements c_i and c_{i+1} are equivalent. So, chain equivalent pairs are those obtainable from one another by a sequence of standard transformations. Also, the relation of being chain equivalent (resp., strictly chain equivalent) is an equivalence relation.

It follows from our definitions that any two chain equivalent pairs $a_i = (x_i, y_i)$ $(i = 1, 2)$ have the same commutator: $[x_1, y_1] = [x_2, y_2]$. Let us put the question the other way around: does the fact that $[x_1, y_1] = [x_2, y_2]$ imply that a_1 and a_2 are chain equivalent? We are going to show that this is, indeed, true "generically", i.e. on a Zariski open set, and this fact will play a crucial role in the proof of Proposition 8.

THEOREM 4: *There exists a $\mathbb{Q}$-defined Zariski open set $V \subset R^4$ with the following property: any $a_i = (x_i, y_i) \in R^2$ $(i = 1, 2)$, such that $[x_1, y_1] = [x_2, y_2]$ and $(x_1, y_1, x_2, y_2) \in V$, are strictly chain equivalent.*

Proof: Fix an integer $t \geq 0$ and consider a subvariety

$$X_t \subset R \times R \times (\mathbf{GL}_n)^{2t}$$

consisting of points $(x, y, a_1, \ldots, a_t, b_1, \ldots, b_t)$ subject to the following conditions:

$$[x, b_1] = 1, \quad [xa_1 \cdots a_i, b_{i+1}] = 1 \quad \text{for } i = 1, \ldots, t-1,$$
$$[a_i, yb_1 \cdots b_i] = 1 \quad \text{and} \quad xa_1 \cdots a_i, \quad yb_1 \cdots b_i \in R$$

for $i = 1, \ldots, t$. Furthermore, define a morphism $\phi_t \colon X_t \to R^4$ by the formula $\phi_t \colon (x, y, a_1, \ldots, a_t, b_1, \ldots, b_t) \to (x, y, xa_1 \cdots a_t, yb_1 \cdots b_t)$. It follows from this definition that if $(x_1, y_1, x_2, y_2) \in \operatorname{Im} \phi_t$, then (x_1, y_1) can be linked with (x_2, y_2) by the following chain of elements:

$$\begin{aligned} &\ldots, (xa_1 \cdots a_i, yb_1 \cdots b_i), \quad (xa_1 \cdots a_i, yb_1 \cdots b_i b_{i+1}), \\ &(xa_1 \cdots a_i a_{i+1}, yb_1 \cdots b_i b_{i+1}), \cdots \end{aligned} \tag{24}$$

Moreover, all pairs in (24) belong to $R \times R$ and each next pair is obtained from the previous one by a standard transformation, i.e. pairs next to each other are equivalent, implying that (x_1, y_1) and (x_2, y_2) are strictly chain equivalent (in particular, $[x_1, y_1] = [x_2, y_2]$). Put

$$Y = \{(x_1, y_1, x_2, y_2) \in R^4 |\, [x_1, y_1] = [x_2, y_2]\}.$$

Then $\phi_t(X_t) \subset Y$, and to prove our Theorem it suffices to show that $\phi_t(X_t)$ is dense in Y, for some t.

LEMMA 9: *X_t and Y are irreducible varieties of dimension $2(n^2+nt)$ and $3n^2+1$, respectively.*

Proof: It is easily seen that the map

$$(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, y_2, x_2)$$

identifies Y with an open subset of $\mathbf{R}_n(\Gamma_2)$, so the assertion regarding Y follows from Theorem 1. The variety X_t can be handled by induction on t. The case $t = 0$ is obvious. To analyse the transition from X_t to X_{t+1}, let us introduce an intermediate variety

$$X'_{t+1} = \{(x, y, a_1, \ldots, a_t, b_1, \ldots, b_t, b_{t+1}) \in X_t \times \mathbf{GL}_n | \\ [xa_1 \cdots a_t, b_{t+1}] = 1 \text{ and } yb_1 \cdots b_t b_{t+1} \in R\}.$$

Since by our construction $xa_1 \cdots a_t \in R$, the equation

$$b_{t+1}(xa_1 \cdots a_t) = (xa_1 \cdots a_t)b_{t+1}$$

for b_{t+1} amounts to a linear system for its coefficients, of rank $n^2 - n$. So, it follows from Lemma 3 that X'_{t+1} is irreducible of dimension $\dim X_t + n$. To pass from X'_{t+1} to X_{t+1} we need to add another component a_{t+1} satisfying

$$a_{t+1}(yb_1 \cdots b_{t+1}) = (yb_1 \cdots b_{t+1})a_{t+1} \quad , \quad xa_1 \cdots a_t a_{t+1} \in R,$$

and repeating the same argument, we conclude that X_{t+1} is irreducible of dimension

$$\dim X_{t+1} = \dim X'_{t+1} + n = \dim X_t + 2n = 2(n^2 + n(t+1)).$$

Lemma 9 is proved. ∎

Now, to prove Theorem 4, it suffices to pick t such that for some $d \in Y$ we have

$$\dim \phi_t^{-1}(d) \leq \dim X_t - \dim Y = 2nt - n^2 - 1.$$

For fixed $(u, v) \in R \times R$ let

$$Z_t(u,v) = X_t \cap ((u,v) \times (\mathbf{GL}_n)^{2t}).$$

LEMMA 10:

(i) *$Z_t(u,v)$ is an irreducible variety of dimension $2nt$;*

(ii) *there exists a pair $(u_0, v_0) \in R \times R$ such that for some t we have*

$$\dim \overline{\phi_t(Z_t(u_0,v_0))} \geq n^2 + 1.$$

Using Lemma 10 it is easy to complete the proof of Theorem 4. Indeed, for any $d \in \phi_t(Z_t(u_0, v_0))$ we have $\phi_t^{-1}(d) \subset Z_t(u_0, v_0)$, so using the dimension values from Lemma 10 we conclude that there must be a point $d \in \phi_t(Z_t(u_0, v_0))$ such that

$$\dim \phi_t^{-1}(d) = \dim Z_t(u_0,v_0) - \dim \overline{\phi_t(Z_t(u_0,v_0))} \leq 2tn - n^2 - 1,$$

as required.

Proof of Lemma 10: (i) is proved in exactly the same fashion as Lemma 9. To prove (ii), let us take

$$u_0 = \operatorname{diag}(1, \rho, \ldots, \rho^{n-1}), \quad v_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

where ρ is a primitive n-th root of unity. It is easy to check that $u_0, v_0 \in R$ and the commutator $[u_0, v_0]$ equals $\xi = \operatorname{diag}(\rho, \ldots, \rho)$. Composing ϕ_t with the projection $Y \to \mathbf{GL}_n \times \mathbf{GL}_n$ onto the second pair of components, we obtain a morphism

$$\rho_t \colon Z_t(u,v) \to W(z), \quad z = [u,v],$$

and we need to prove that for some t

$$\dim \overline{\rho_t(Z_t(u_0,v_0))} \geq n^2 + 1.$$

It easily follows from our definitions that

$$(25) \qquad \cdots \subset \rho_t(Z_t(u_0,v_0)) \subset \rho_{t+1}(Z_{t+1}(u_0,v_0)) \subset \cdots$$

Pick t such that the dimension of $L = \overline{\rho_t(Z_t(u_0, v_0))}$ is maximal possible. Then by (25), $\rho_s(Z_s(u_0, v_0)) \subset L$ for any s. However the union $\cup_{s\geq 0} \rho_s(Z_s(u_0, v_0))$ coincides with the equivalence class $E = [(u_0, v_0)]$ with respect to the strict chain equivalence. So, all that remains to be proven is that $\dim \bar{E} \geq n^2 + 1$.

Since ξ belongs to the center of $\mathbf{GL}_n$, this group acts on $W(\xi)$ by conjugation. Besides, for any point $(u, v) \in W(\xi)$ and any $g \in \mathbf{GL}_n$ we obviously have $g[u, v] = [(gug^{-1}, gvg^{-1})]$. This implies, in particular, that E is invariant under $Z(u_0)$ and $Z(v_0)$, and therefore also under the subgroup H generated by them.

LEMMA 11: $H = \mathbf{GL}_n$.

Proof: Clearly, $Z(u_0)$ coincides with the diagonal torus S, so H is a connected subgroup of $\mathbf{GL}_n$ (as generated by two connected subgroups, cf. [Bo]) and contains S and v_0. Let Φ be the set of roots of H with respect to S. Each $\alpha \in \Phi$ can be identified with a pair (i, j) where i and j are distinct residue classes mod n. Direct computation shows that

$$v_0 \operatorname{diag}(s_1, \dots, s_n) v_0^{-1} = \operatorname{diag}(s_n, s_1, \dots, s_{n-1}),$$

therefore v_0 acts on Φ as follows: $v_0(i, j) = (i + 1, j + 1)$. Besides, from the commutator relations we derive that if $(i, j), (j, k) \in \Phi$ and $i \neq k$ then $(i, k) \in \Phi$. Let r be the least positive integer such that $(0, r) \in \Phi$.

Using Euclid's algorithm it is easy to show that r divides n and for any $(i, j) \in \Phi$ the difference $i - j$ is divisible by r. Then any root $\alpha \in \Phi$ restricts trivially to the r-dimensional subtorus consisting of matrices of the form

$$\operatorname{diag}(t_1, t_2, \dots, t_r, t_1, t_2, \dots, t_r, t_1, t_2, \dots, t_r),$$

consequently, these matrices belong to the center of H.

However, a diagonal matrix which commutes with v_0 is necessarily a scalar one, forcing $r = 1$. So, $(0, 1) \in \Phi$, easily implying that $(i, j) \in \Phi$ for any $i \neq j$, hence $H = \mathbf{GL}_n$. Lemma 11 is proved. ∎

It follows from Lemma 11 that the image of the morphism

$$\delta \colon \mathbf{G}_m \times \mathbf{G}_m \times \mathbf{GL}_n \to \mathbf{GL}_n \times \mathbf{GL}_n, \ (\alpha, \beta, h) \mapsto (h(\alpha u_0)h^{-1}, h(\beta v_0)h^{-1}),$$

is contained in E, and to complete the proof we need to show only that for any $l = (\alpha, \beta, h)$ we have $\dim \delta^{-1}(\delta(l)) = 1$. Let $l_1 = (\alpha_1, \beta_1, h_1) \in \delta^{-1}(\delta(l))$ and

M be an irreducible component of $\delta^{-1}(\delta(l))$ containing l_1. Since $u_0^n = v_0^n = E_n$, we have $\alpha_1^n = \alpha^n, \beta_1^n = \beta^n$. So, there are only finitely many possibilities for α_1, β_1 and therefore for any $l' = (\alpha', \beta', h') \in M$ we have $\alpha' = \alpha_1, \beta' = \beta_1$. Then $h_1^{-1}h' \in Z(u_0) \cap Z(v_0) =$ (scalar matrices), and $\dim M = 1$, as required. Theorem 4 is proved. ∎

The fact (actually) proved in Theorem 4 can be reformulated as follows: there exist an integer $t > 0$ and Zariski open set $C \subset R \times R$ such that for $(u, v) \in C$ the morphism $\rho_t\colon Z_t(u, v) \to W(z)$, $z = [u, v]$, is dominant. It means that almost all the points on $W(z)$ are strictly chain equivalent. It is natural to expect the sharper result that for any z in some Zariski open subset of $\mathbf{SL}_n$ all points $(x, y) \in W(z)$ with regular x and y are (strictly) chain equivalent (one can show that it is, indeed, the case for $z = \xi$).

Let us point out one obstruction to the surjectivity of ρ_t: it is easy to see that for any $(u', v') \in \rho_t(Z_t(u, v))$ we have $Z(u) \cap Z(v) = Z(u') \cap Z(v')$; so points $(u_1, v_1), (u_2, v_2) \in W(z)$ such that $Z(u_1) \cap Z(v_1) \neq Z(u_2) \cap Z(v_2)$ cannot be strictly chain equivalent (one can give examples when $\dim Z(u_1) \cap Z(v_1) = 1$ and $\dim Z(u_2) \cap Z(v_2) > 1$). We would like to put forward a conjecture that there should not be other obstructions to ρ_t being dominant, viz. There exists an integer $t > 0$ such that for any $(u, v) \in R \times R$ with the properties: $\dim Z(u) \cap Z(v) = 1$ and the commutator variety $W(z)$, $z = [u, v]$, is irreducible of dimension $n^2 + 1$, the morphism $\rho_t\colon Z_t(u, v) \to W(z)$ should be dominant. One can show that if $z \in R$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and none of the products $\lambda_{i_1} \cdots \lambda_{i_l}$ is 1 (note that this condition describes a Zariski open set) then for any $(u, v) \in W(z)$ we already have $\dim Z(u) \cap Z(v) = 1$. Now, assuming our conjecture and picking z such that in addition $W(z)$ is irreducible of dimension $n^2 + 1$, for any two regular points $(u_i, v_i) \in W(z)$ $(i = 1, 2)$ we would have

$$\rho_t(Z_t(u_1, v_1)) \cap \rho_t(Z_t(u_2, v_2)) \neq \emptyset,$$

implying strict chain equivalence of (u_1, v_1) and (u_2, v_2).

We conclude this section with the proof of Proposition 8. Put $R' = R \cap \mathbf{SL}_n$, and for $(u, v) \in R' \times R'$ let

$$Z'_t(u, v) = Z_t(u, v) \cap ((u, v) \times (\mathbf{SL}_n)^{2t}).$$

Clearly, the restriction of ρ_t to $Z'_t(u, v)$ defines a morphism $\rho'_t\colon Z'_t(u, v) \to$

$W'(z) = W(z) \cap (\mathbf{SL}_n \times \mathbf{SL}_n)$, and ρ'_t is dominant if ρ_t is such (and $W'(z)$ is irreducible).

LEMMA 12:

(i) *For any $t \geq 0$, any extension $K/\mathbb{Q}$ and any $(u,v) \in R'_K \times R'_K$ the variety $Z'_t(u,v)$ is irreducible and K-unirational.*

(ii) *There exist an integer $t > 0$ and a $\mathbb{Q}$-defined Zariski open set $C' \subset R' \times R'$ such that for $(u,v) \in C'$ the morphism ρ'_t: $Z'_t(u,v) \to W'(z)$ is dominant.*

Proof: (ii) immediately follows from the discussion above. Let us prove (i) by induction on t. The case $t = 0$ is obvious while the transition from $Z'_t(u,v)$ to $Z'_{t+1}(u,v)$ results in adding another pair of components. Let us introduce an intermediate variety $M'_{t+1}(u,v)$ obtained by adding just one component and show that at each step, i.e. going from Z'_t to M'_{t+1} and from M'_{t+1} to Z'_{t+1}, we end up with an irreducible K-unirational variety. So, let

$$M_{t+1}(u,v) = \{(u,v,a_1,\ldots,a_t,b_1,\ldots,b_t,b_{t+1}) \in Z'_t(u,v) \times \mathbf{GL}_n| \\ [ua_1\cdots a_t, b_{t+1}] = 1, vb_1\cdots b_t b_{t+1} \in R\}$$

and M'_{t+1} be the subvariety of M_{t+1} defined by the condition $b_{t+1} \in \mathbf{SL}_n$. Then as in the proof of Lemma 9, we obtain from Lemma 3 that M_{t+1} is irreducible. After that the irreducibility of M'_{t+1} can be proved by repeating the argument used in Proposition 6. Finally, if $\omega = (u,v,\bar{a}_1,\ldots,\bar{a}_t,\bar{b}_1,\ldots,\bar{b}_t)$ is a generic point of $Z'_t(u,v)$ over K and $L = K(\omega)$ then $K(M'_{t+1}(u,v))$ can be identified with $L(C)$ where C is a L-torus equal to $Z_{\mathbf{SL}_n}(u\bar{a}_1\cdots\bar{a}_t)$. However, any torus is unirational over the field of definition [Bo] and L is unirational over K by the induction hypotheses implying K-unirationality of M'_{t+1}. The transition from M'_{t+1} to Z'_{t+1} is handled in exactly the same way. Lemma 12 is proved. ∎

Lemma 12 implies that for any $(u,v) \in C'_K$ the corresponding commutator variety $W'(z)$, $z = [u,v]$, is K-unirational. So, to complete the proof of Proposition 8 we need to show that there exists a $\mathbb{Q}$-defined Zariski open set $V \subset \mathbf{SL}_n$ such that, for any $z \in V_K$, $W'(z)$ contains a point $(u,v) \in C'_K$. This follows from the statement below which is interesting in its own right.

PROPOSITION 9: *Let $D \subset \mathbf{SL}_n \times \mathbf{SL}_n$ be a $\mathbb{Q}$-defined Zariski open set. Then there exists a $\mathbb{Q}$-defined rational map θ: $\mathbf{SL}_n \to \mathbf{SL}_n \times \mathbf{SL}_n$ such that for any z from the domain of θ we have $\theta(z) \in W'(z)$ and $\operatorname{Im}\theta \cap D \neq \emptyset$.*

Proof: Let $T \subset \mathbf{SL}_n$ be a diagonal torus and for $a = \mathrm{diag}(a_1, \ldots, a_n)$ let

$$a(x_2, \ldots, x_n) = \begin{pmatrix} a_1 & & & 0 \\ x_2 & a_2 & & \\ \vdots & 0 & \ddots & \\ x_n & & & a_n \end{pmatrix}.$$

CLAIM: *There exists a Zariski $\mathbb{Q}$-open set $T_0 \subset T$ such that for $a \in T_0$ and any z in some (non-empty) Zariski open subset $B_0 \subset \mathbf{SL}_n$ the system*

$$\sigma_i(za(x_2, \ldots, x_n)) = \sigma_i(a(x_2, \ldots, x_n), \quad i = 1, \ldots, n-1 \tag{26}$$

(where σ_i as in §2 is the coefficient of λ^{n-i} in the characteristic polynomial) has a unique solution $(x_2(z), \ldots, x_n(z))$.

Indeed, it follows from §2 that (26) amounts to a linear system for $x_2, \ldots, x_n$:

$$\begin{cases} A_{12}x_2 + \cdots + A_{1n}x_n & = A_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ A_{n-12}x_2 + \cdots + A_{n-1n}x_n & = A_{n-1} \end{cases} \tag{27}$$

in which $A_{ij} = A_{ij}(a, z), A_i = A_i(a, z)$. Now, direct computations show that for $z = z_0$ where

$$z_0 = \begin{pmatrix} 1 & \ldots & 1 \\ & \ddots & 0 \\ 0 & & 1 \end{pmatrix}$$

we have $A_{12} = \cdots = A_{1n} = 1$ and for $i > 1$

$$A_{ij} = \sum_{\substack{2 \le k_1 < k_{i-1} \le n \\ k_s \ne j}} a_{k_1} \cdots a_{k_i}.$$

Taking $a_2 = 1, a_3 = \lambda, \ldots, a_n = \lambda^{n-2}$ it is easy to ascertain that the determinant $d(a_2, \ldots, a_n)$ of the matrix of (27) for $z = z_0$ is not identically zero. Let T_0 be defined by the condition $d(a_2, \ldots, a_n) \neq 0$. Then for B_0 corresponding to $(a_1, \ldots, a_n) \in T_0$ one can take the set of z such that the matrix of (27) is non-degenerate. (Note that by our construction $z_0 \in B_0$, so B_0 is non-empty, and moreover, B_0 is $\mathbf{Q}$-defined if $a \in T_{\mathbf{Q}}$.)

Applying conjugation, if necessary, we may assume that the projection $\mathrm{pr}_2(D)$ of D onto the second component meets T, so we can pick a regular $a \in (T_0)_{\mathbb{Q}} \cap \mathrm{pr}_2(D)$. Furthermore, we can find non-zero $x_2^0, \ldots, x_n^0 \in \mathbb{Q}$, $b_0 \in \mathbf{SL}_n(\mathbb{Q})$

such that $(b_0, a_0 = a(x_2^0, \ldots, x_n^0)) \in D$ and $[b_0, a_0] \in B_0$. Since $C = Z_{\mathbf{SL}_n}(a_0)$ is a $\mathbb{Q}$-split torus, the morphism $\mathbf{SL}_n \xrightarrow{\alpha} \mathbf{SL}_n/C$ admits a $\mathbf{Q}$-defined rational section, i.e. there is a $\mathbb{Q}$-defined subvariety $F \subset \mathbf{SL}_n$ such that $\alpha|_F$ is a birational isomorphism onto $\mathbf{SL}_n/C$, and we may assume that $b_0 \in F$. Now taking into account that

$$\mathrm{diag}(1, x_2, \ldots, x_n)\ a_0\ \mathrm{diag}(1, x_2, \ldots, x_n)^{-1} = a(x_2^0 x_2, \ldots, x_n^0 x_n)$$

we conclude from the above that the morphism

$$\rho\colon F \times \mathbf{G}_m^{n-1} \to \mathbf{SL}_n$$

defined by

$$\rho\colon (f, x_2, \ldots, x_n) \mapsto \mathrm{diag}(1, x_2, \ldots, x_n)\ [f, a_0]\ \mathrm{diag}(1, x_2, \ldots, x_n)^{-1}$$

is a birational isomorphism. Let its inverse $\rho^{-1}\colon \mathbf{SL}_n \to F \times \mathbf{G}_m^{n-1}$ have the following components:

$$\rho^{-1}(z) = (\rho_0(z), \rho_2(z), \ldots, \rho_n(z)).$$

Put $\mu(z) = \mathrm{diag}\ (1, \rho_2(z), \ldots, \rho_n(z))$. Then the required map $\theta\colon \mathbf{SL}_n \to \mathbf{SL}_n \times \mathbf{SL}_n$ can be defined as follows:

$$\theta(z) = (\mu(z)\rho_0(z)\mu(z)^{-1}, \mu(z)a_0\mu(z)^{-1}).$$

Proposition 9 is proved. ∎

References

[Ba-Lu] H. Bass and A. Lubotzky, *Automorphisms of groups and of schemes of finite type*, Israel Journal of Mathematics **44** (1983), 1–22.

[Bo] A. Borel, *Linear Algebraic Groups*, Springer-Verlag, Berlin–Heidelberg–New York, 1991.

[B-LeB] Ch. Bessenrodt and L. Le Bruyn, *Stable rationality of certain* $\mathbf{PGL}_n$*-quotients*, Inventiones mathematicae **104** (1991), 179–199.

[C] C. Chevalley, *On algebraic group varieties*, Journal of the Mathematical Society of Japan **6** (1954), 303–324.

[Co-Sa] J.-L. Colliot-Thelene and J.-J. Sansuc, *La R-equivalence sur les tores*, Annales Scientifiques de l'École Normale Supérieure **10** (1977), 175–229.

[D] D. Djokovič, *Cosets of centralizers in the general linear groups contain regular elements*, to appear in: Linear Algebra and its Applications.

[F1] E. Formanek, *The center of the ring of* 3 *by* 3 *generic matrices*, Linear and Multilinear Algebra **7** (1979), 203–212.

[F2] E. Formanek, *The center of the ring of* 4 *by* 4 *generic matrices*, Journal of Algebra **62** (1980), 304–319.

[G] M. Gerstenhaber, *On dominance and varieties of commuting matrices*, Annals of Mathematics **73** (1961), 324–348.

[Go] W. Goldman, *Topological components of spaces of representations*, Inventiones mathematicae **93** (1988), 557–607.

[Gu] R. M. Guralnick, *A note on commuting pairs of matrices*, Linear and Multilinear Algebra **31** (1992), 71–75.

[J] Jun Li, *The space of surface group representations*, Preprint UCLA.

[K] V. V. Kursov, *Commutators in multiplicative groups of skew-fields*, Doklady Belorussian Academy of Science **26** (1982), 101–103 (in Russian).

[La] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Addison Wesley-Benjamin, Reading, Massachusets, 1973.

[LeB] L. Le Bruyn, *Centers of generic division algebras, the rationality problem 1965–1990*, Israel Journal of Mathematics **76** (1991), 97–111.

[Lu-M] A. Lubotzky and A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the American Mathematical Society **58** (1985), 1–116.

[L-S] R. Lyndon and P. Shupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[Mo-Ta] T. Motzkin and O. Taussky, *Pairs of matrices with property L. II*, Transactions of the American Mathematical Society **80** (1955), 387–401.

[P] R. S. Pierce, *Associative Algebras*, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[Pl] V. P. Platonov, *Rings and varieties of representations of finitely generated groups*, Topics in Algebra. Minsk **4** (1989), 36–40 (in Russian).

[Pl-R] V. P. Platonov and A. S. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, New York, 1993.

[R-Be] A. S. Rapinchuk and V. V. Benyash-Krivetz, *Geometric representation theory for fundamental groups of compact orientable surfaces*, Doklady Russian Academy of Science **329** (1993), 140–143 (in Russian).

[R-Be-Ch] A. S. Rapinchuk, V. V. Benyash-Krivetz and V. I. Chernousov, *Representation varieties of the fundamental groups of compact orientable surfaces*, preprint 94-030 of the SFB-343 "Diskrete Strukturen in der Mathematik" (Universität Bielefeld, Germany).

[Ri] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Mathematica **38** (1979), 311–327.

[Sh1] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.

[Sh2] I. R. Shafarevich, *Algebraic surfaces*, Proceedings of the Steklov Institute of Mathematics **75** (1965) (transl. by the AMS, 1967).

[Sp-St] T. Springer and R. Steinberg, *Conjugacy classes*, in *Seminar in Algebraic groups and Related Finite Groups* (ed. A. Borel et al.), Lecture Notes in Mathematics **131**, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[T] R. G. Thompson, *Commutators in the special and general linear groups*, Transactions of the American Mathematical Society **101** (1961), 16–33.

[V] V. E. Voskresenskii, *Algebraic Tori*, Nauka, Moscow, 1977 (in Russian).