INCREMENT OF THE OBJECTIVE FUNCTION
AND OPTIMALITY CRITERION FOR ONE
NON-HOMOGENEOUS NETWORK FLOW
PROGRAMMING PROBLEM

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Abstract: For an linear non-homogeneous flow programming problem with
additional constraints of general kind are obtained the increment of the objective
function using network properties of the problem and principles of decom-
position of a support. Optimality conditions are received.

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Consider the following mathematical model of inhomogeneous extreme network flow problem

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^k \rightarrow \min, \quad (1)$$

$$\sum_{j \in I^+_i(U^k)} x_{ij}^k - \sum_{j \in I^-_i(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K; \quad (2)$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = 1, l; \quad (3)$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, \quad \text{for } (i,j) \in U_0; \quad (4)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, \quad \text{for } k \in K_1(i,j), (i,j) \in U; \quad (5)$$

$$x_{ij}^k \geq 0, \quad \text{for } k \in K(i,j) \setminus K_1(i,j), (i,j) \in U, \quad (6)$$

where $G = (I,U)$ – a finite orientated connected network without multiple arcs and loops, $I$ is a set of nodes and $U \subseteq I \times I$ is a set of arcs; $K = \{1, \ldots, |K|\}$ – a finite non-empty set of different products (commodities) is transported through the network $G$. For each $k \in K$ there exists a connected subnetwork $G^k = (I^k, U^k) \subseteq G$, such that $U^k \subseteq U$ is a non-empty set of arcs carrying the $k$-th product, $I^k = I(U^k)$ – is the set of nodes used for transporting (producing/consuming/transiting) the $k$-th product. In order to distinguish the products, which can simultaneously pass through an arc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j) \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$. Now let us define a set $U_0 \subseteq U$ as an arbitrary subset of multiarcs of the network $G$. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j), |K_0(i,j)| > 1$. For all arcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be non-negative. Moreover, each arc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the arc $(i,j)$. $I^+_i(U^k) = \{j \in I^k : (i,j) \in U^k\}, I^-_i(U^k) = \{j \in I^k : (j,i) \in U^k\}; x_{ij}^k$ – amount of the $k$-th product transported through an arc $(i,j)$; $c_{ij}^k$ – transportation cost through an arc $(i,j)$ of a unit of the $k$-th product; $d_{ij}^k$ – carrying capacity of an arc $(i,j)$ for the $k$-th product; $d_{ij}^0$ – aggregate capacity of an arc $(i,j) \in U_0$ for a total amount of
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products $K_0(i,j)$; $\lambda_{ij}^{kp}$ – weight of a unit of the $k$-th product transported through an arc $(i,j)$ in the $p$-th additional constraint; $\alpha_p$ – total weighted amount of products imposed by the $p$-th additional constraint; $a_i^k$ – intensity of a node $i$ for the $k$-th product.

2. Formula for Increment of the Objective Function

Let $x = (x_{ij}^k, (i,j) \in U, k \in K(i,j))$ be a plan [2] of the problem (1)-(6), i.e. components of the vector $x$ meet the conditions (2)-(6). Along with the plan $x$ let us define support plan $\{x, U_S\}$ as a pair, containing of an arbitrary flow $x$ and a support $U_S = \{U_k^S, k \in K; U_S \subseteq U_0, U_0 = \{(i,j) \in U : |K_0^S(i,j)| > 1\}$ of the problem (1)-(6) [2, 4]. Let us consider some other plan $x = (x_{ij}^k + \Delta x_{ij}^k, (i,j) \in U, k \in K(i,j))$. Then $\Delta x = (\Delta x_{ij}^k, (i,j) \in U, k \in K(i,j))$ is the vector of flow increments along the arc $(i,j) \in U$.

Let us denote

$$z_{ij} = \sum_{k \in K(i,j)} x_{ij}^k, \quad \bar{z}_{ij} = \sum_{k \in K(i,j)} \bar{x}_{ij}^k,$$

$$\Delta z_{ij} = \Delta z_{ij} = \sum_{k \in K(i,j)} \Delta x_{ij}^k, (i,j) \in U_0.$$  \hspace{1cm} (7)

Since the plan $\bar{x}$ meets the conditions (2)-(6) then the following relations are true

$$\sum_{j \in I^+_k(U^k)} \bar{x}_{ij}^k - \sum_{j \in I^-_k(U^k)} \bar{x}_{ji}^k = a_i^k, \quad i \in I^k, \ k \in K,$$ \hspace{1cm} (8)

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} \bar{x}_{ij}^k = \alpha_p, \quad p = 1, l,$$ \hspace{1cm} (9)

$$\sum_{k \in K_0(i,j)} \bar{x}_{ij}^k \leq d_{ij}^0, \quad \bar{x}_{ij}^k \geq 0, \quad k \in K_0(i,j), \ (i,j) \in U^*,$$ \hspace{1cm} (10)

where the constraints (4) are written down only for the support multiarcs $U^*$.

Subtracting from (8)-(10) the corresponding constraints (2)-(4), we obtain:

$$\sum_{j \in I^+_k(U^k)} \Delta x_{ij}^k - \sum_{j \in I^-_k(U^k)} \Delta x_{ji}^k = 0, \quad i \in I^k, \ k \in K,$$ \hspace{1cm} (11)

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} \Delta x_{ij}^k = 0, \quad p = 1, l,$$ \hspace{1cm} (12)
where $\Delta z_{ij}$ is defined by formula (7).

Let us order components of solution of system (11)-(13) the following way:

$$\Delta x'_T = (\Delta x'^{k}_i, \Delta x'^{k}_j, \Delta x'_N), \quad \Delta x'^{k}_i = (\Delta x'^{k}_i(i, j) \in U^k_T, k \in K), \quad \Delta x'^{k}_j = (\Delta x'^{k}_j(i, j) \in U^k_C, k \in K), \quad \Delta x'_N = (\Delta x'^{k}_N(i, j) \in U^k_N, k \in K), \quad U^k_T = U^k \setminus (U^k_B \cup U^k_C), \quad U^k_B - \text{spanning tree of the graph } G^k, \quad k \in K.$

The general solution of the homogeneous system (11) is the following [4]:

$$\Delta x^{k}_ij = \sum_{(\tau, \rho) \in U^k \setminus U^k_T} \Delta x^{k}_{i,j} \sigma(i, j)^{L^k_{(\tau, \rho)}}, (i, j)^k \in U^k_T, k \in K, \quad \text{where} \quad \Delta x^{k}_{i,j} = \sum_{k \in K} \sum_{(i, j) \in U^k} c^{k}_{ij} \Delta x^{k}_{ij}.$$

Let us put the items, corresponding to components of the vector $\Delta x'^{k}_T$, together:

$$\Delta \varphi(x) = \sum_{k \in K} \sum_{(i, j) \in U^k} c^{k}_{ij} \Delta x^{k}_{ij} = \sum_{k \in K} \sum_{(i, j) \in U^k_T} \Delta x^{k}_{ij} + \sum_{k \in K} \sum_{(i, j) \in U^k \setminus U^k_T} c^{k}_{ij} \Delta x^{k}_{ij}. \quad (15)$$

Let us substitute (14) into (15):

$$\Delta \varphi(x) = \sum_{k \in K} \sum_{(i, j) \in U^k} c^{k}_{ij} \left[ \sum_{(\tau, \rho) \in U^k \setminus U^k_T} \Delta x^{k}_{i,j} \sigma(i, j)^{L^k_{(\tau, \rho)}} \right] + \sum_{k \in K} \sum_{(i, j) \in U^k \setminus U^k_T} c^{k}_{ij} \Delta x^{k}_{i,j} \sigma(i, j)^{L^k_{(\tau, \rho)}}$$

$$= \sum_{k \in K} \sum_{(\tau, \rho) \in U^k \setminus U^k_T} \left[ c^{k}_{ij} \Delta x^{k}_{i,j} \sigma(i, j)^{L^k_{(\tau, \rho)}} \right] + \sum_{(i, j) \in U^k \setminus U^k_T} c^{k}_{ij} \Delta x^{k}_{i,j} \sigma(i, j)^{L^k_{(\tau, \rho)}}.$$

Let us denote $\sum_{(i, j) \in L^k_{(\tau, \rho)}} c^{k}_{ij} \sigma(i, j)^{L^k_{(\tau, \rho)}},$ with $\Delta^{k}_{\tau, \rho}$. Then

$$\Delta \varphi(x) = \sum_{k \in K} \sum_{(\tau, \rho) \in U^k \setminus U^k_T} \Delta^{k}_{\tau, \rho} \Delta x^{k}_{i,j}. \quad (16)$$
Knowing that $U_k \setminus U_p^k = U_p^k \cup U_N$, we break the sum again:

$$\Delta \varphi(x) = \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^C} \tilde{\Delta}^k_{\tau \rho} \Delta x^k_{\tau \rho} + \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^N} \tilde{\Delta}^k_{\tau \rho} \Delta x^k_{\tau \rho}. \quad (17)$$

By analogy with [6], [7] we obtain the components of the vector $\Delta x_C^\prime$ for system (11)-(13):

$$\Delta x^k_{\tau \rho} = \sum_{p=1}^l \nu(l(\tau, \rho)^k, p) \tilde{\beta}^k_p + \sum_{(i,j) \in U^*} \nu(l(\tau, \rho)^k, i+\xi(i,j)) \tilde{\beta}^k_{l+\xi(i,j)}, \quad (18)$$

The values of the components of the vectors $\tilde{\beta}^k_p$ and $\tilde{\beta}^k_{\xi(i,j)}$ are computed according to the following formulas:

$$\tilde{\beta}^k_p = -\sum_{k \in K} \sum_{(\tau, \rho) \in U_k^C} R_p(L^k_{l(\tau, \rho)}) \Delta x^k_{\tau \rho}, \quad p = 1, l, \quad (19)$$

$$\tilde{\beta}^k_{\xi(i,j)} = \Delta z_{ij} - \sum_{k \in K_0(i,j)} \sum_{(\tau, \rho) \in U_k^F} \delta_{\xi(i,j)}(L^k_{l(\tau, \rho)}) \Delta x^k_{\tau \rho}, \quad (i,j) \in U^*. \quad (20)$$

Taking into account the formula (14) we obtain:

$$\Delta \varphi(x) = \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^C} \tilde{\Delta}^k_{\tau \rho} \left[ \sum_{p=1}^l \nu(l(\tau, \rho)^k, p) \tilde{\beta}^k_p \right. \left. + \sum_{(i,j) \in U^*} \nu(l(\tau, \rho)^k, i+\xi(i,j)) \tilde{\beta}^k_{l+\xi(i,j)} \right] + \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^N} \tilde{\Delta}^k_{\tau \rho} \Delta x^k_{\tau \rho}. \quad (21)$$

Let us introduce the following denotations:

$$r_p = \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^C} \tilde{\Delta}^k_{\tau \rho} \nu(l(\tau, \rho)^k, p), \quad p = 1, l,$$

$$r_{ij} = \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^F} \tilde{\Delta}^k_{\tau \rho} \nu(l(\tau, \rho)^k, i+\xi(i,j)), \quad (i,j) \in U^*. \quad (22)$$

Taking into account the denotations made, we may represent $\Delta \varphi(x)$ the following way:

$$\Delta \varphi(x) = \sum_{(i,j) \in U^*} r_{ij} \Delta z_{ij} + \sum_{k \in K} \sum_{(\tau, \rho) \in U_k^C} \tilde{\Delta}^k_{\tau \rho} \Delta x^k_{\tau \rho}.$$
\[- \sum_{p=1}^{l} r_p R_p(L_{i(\tau,\rho)}^k) - \sum_{(i,j) \in U^*} r_{ij} \delta_{(i,j)}(L_{i(\tau,\rho)}^k) \] 
\[ \Delta x_{i(\tau,\rho)}^k \]
\[ = \sum_{(i,j) \in U^*} \gamma_{ij} \Delta z_{ij} + \sum_{k \in K} \sum_{(\tau,\rho) \in U_k^k} \Delta x_{i(\tau,\rho)}^k, \tag{22} \]

where $\Delta z_{ij}$ is defined by formula (7),
\[ \Delta z_{ij} = \tilde{\Delta} z_{ij} - \sum_{p=1}^{l} r_p R_p(L_{i(\tau,\rho)}^k) - \sum_{(i,j) \in U^*} r_{ij} \delta_{(i,j)}(L_{i(\tau,\rho)}^k), \tag{23} \]

3. Conditions of Optimality

**Definition 1.** A support plan $\{x, U_S\}$ is called nonsingular if the following conditions are met:
\[ 0 < x_{ij}^k < d_{ij}^k, \quad k \in K_S^1(i,j), \quad (i,j) \in U, \]
\[ x_{ij}^k > 0, \quad k \in K_S^0(i,j), \quad (i,j) \in U_0, \]
\[ x_{ij}^k > 0, \quad k \in K_S(i,j) \setminus K_S^1(i,j), \quad (i,j) \in U \setminus U_0, \]
\[ 0 < \sum_{k \in K_0(i,j)} x_{ij}^k < d_{ij}^0, \quad (i,j) \in U_0 \setminus U^* . \tag{24} \]

**Theorem 1.** Let $\{x, U_S\}$ be a support plan. The following conditions are necessary for optimality of $\{x, U_S\}$ and are also sufficient if $\{x, U_S\}$ is nonsingular:
\[ x_{ij}^k = 0 \quad \text{if} \quad \Delta_{ij}^k < 0, \]
\[ x_{ij}^k = d_{ij}^k \quad \text{if} \quad \Delta_{ij}^k < 0, \]
\[ x_{ij}^k \in [0, d_{ij}^k] \quad \text{if} \quad \Delta_{ij}^k = 0, k \in K_S^1(i,j), (i,j) \in U; \]
\[ x_{ij}^k = 0 \quad \text{if} \quad \Delta_{ij}^k > 0, \]
\[ x_{ij}^k \geq 0 \quad \text{if} \quad \Delta_{ij}^k = 0, k \in K_S^0(i,j), (i,j) \in U_0; \tag{26} \]
\[ x_{ij}^k = 0 \quad \text{if} \quad \Delta_{ij}^k > 0, \]
\[ x_{ij}^k \geq 0 \quad \text{if} \quad \Delta_{ij}^k = 0, k \in K_N(i,j) \setminus K_N^1(i,j), (i,j) \in U \setminus U_0; \tag{27} \]
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\[ \sum_{k \in K_{0(i,j)}} x_{ij}^k = \begin{cases} 0 & \text{if } \gamma_{ij} > 0, \\ d_{ij}^0 & \text{if } \gamma_{ij} < 0, \\ [0, d_{ij}^0] & \text{if } \gamma_{ij} = 0, \end{cases} \tag{28} \]

Proof. The proof is given in [4].

In the criterion of an optimality (25)–(28) we used the analytical formula for computing reduced costs \( \Delta^k_{\tau \rho} \):

\[ \Delta^k_{\tau \rho} = \tilde{\Delta}^k_{\tau \rho} - \sum_{p=1}^{l} r_p R_p(L^k_{l(\tau, \rho)}) - \sum_{(i, j) \in U^*} r_{ij} \delta \xi_{(i, j)}(L^k_{l(\tau, \rho)}), \]

\[ (\tau, \rho)^k \in U^k_N, \quad k \in K, \quad \gamma_{ij} = r_{ij}. \]

For computing reduced costs \( \Delta^k_{\tau \rho} \) we can build the vector \( r = (r_p : p = 1, \ldots, l; \gamma_{ij}, (i, j) \in U^*) \), \( u_i = (u^k_i, k \in K(i)), i \in I \) as a solution of the potential system [2, 4].

We compute the reduced costs \( \Delta^k_{ij} \) for the arcs \( (i, j)^k \in U^k_N, U^N_k = U^k \setminus U^k_S, \)

\[ k \in K \] and for the arcs \( (i, j)^k, \quad k \in K^0_S(i, j), \quad (i, j) \in U^* \) using the following formula:

\[ \Delta^k_{ij} = c^k_{ij} - \left( u^k_i - u^k_j + \sum_{p=1}^{l} \lambda^k_{ij} r_p \right). \tag{29} \]

One may check that the formulas (23) and (29) give the identical results for the problem (1)-(6). Strategy of application (23) or (29) are described in [1, 4, 6].

References


