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DECOMPOSITION OF THE NETWORK SUPPORT FOR ONE NON-HOMOGENEOUS NETWORK FLOW PROGRAMMING PROBLEM

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Abstract: We consider one extremal linear non-homogeneous problem of flow programming with additional constraints of general kind. We use the network properties of the non-homogeneous problem for the decomposition of a network support into trees and cyclic parts.

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1. Mathematical Model

Let $G = (I, U)$ be a finite oriented connected network without multiple arcs and loops, where I is a set of nodes and $U \subset I \times I$ is a set of arcs. Assume that a finite non-empty set $K = \{1, \ldots, |K|\}$ of different products (commodities) is

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transported through the network G. For each $k \in K$ there exists a connected subnetwork $G^k = (I^k, U^k) \subseteq G$, such that $U^k \subseteq U$ is a non-empty set of arcs carrying the k-th product, $I^k = I(U^k)$, $I(U^k)$ – is the set of nodes used for transporting (producing/consuming/transiting) the k-th product. In order to distinguish the products, which can simultaneously pass through an arc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j) \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$.

Now let us define a set U_0 as an arbitrary subset of multiarcs of the network G, $U_0 \subseteq U$. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j), |K_0(i,j)| > 1$. For all arcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be non-negative. Moreover, each arc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the arc (i,j).

Thus, given transportation costs through arcs $(i,j) \in U$ for all products, we consider the min-cost flow problem on the described network G, where the products are, additionally, related by equality constraints of a general kind.

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} c_{ij}^k x_{ij}^k \longrightarrow \min, \tag{1}$$

$$\sum_{j\in I_i^+(U^k)} x_{ij}^k - \sum_{j\in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K; \tag{2}$$

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = \overline{1,l}; \tag{3}$$

$$\sum_{k\in K_0(i,j)} x_{ij}^k \le d_{ij}^0, \quad \text{for } (i,j) \in U_0; \tag{4}$$

$$0 \le x_{ij}^k \le d_{ij}^k, \quad \text{for } k \in K_1(i,j), (i,j) \in U; \tag{5}$$

$$x_{ij}^k \ge 0, \quad \text{for } k \in K(i,j) \setminus K_1(i,j), (i,j) \in U, \tag{6}$$

where $I_i^+(U^k) = \{j \in I^k : (i,j) \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j,i) \in U^k\}$; x_{ij}^k – amount of the k-th product transported through an arc (i,j); c_{ij}^k – transportation cost through an arc (i,j) of a unit of the k-th product; d_{ij}^k – carrying capacity of an arc (i,j) for the k-th product; d_{ij}^0 – aggregate capacity of an arc $(i,j) \in U_0$ for a total amount of products $K_0(i,j)$; λ_{ij}^{kp} – weight of a unit of the k-th product transported through an arc (i,j) in the p-th additional constraint; α_p – total weighted amount of products imposed by the p-th additional constraint; a_i^k – intensity of a node i for the k-th product.

Definition 1. The vector $x = (x_{ij}^k, (i,j) \in U, k \in K(i,j))$ is a non-homogeneous flow on the network G, if it satisfies the constraints (2)-(6).

For brevity, further in the paper, we will call a "non-homogeneous flow", simply, "a flow" and a "multinetwork", simply "a network".

Definition 2. The flow $x^0 = (x_{ij}^{0k}, (i,j) \in U, k \in K(i,j)), x^0 \in X$, where X is a set of all flows, is optimal if

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} c_{ij}^k x_{ij}^{0k} = \min_{x\in X} \sum_{(i,j)\in U} \sum_{k\in K(i,j)} c_{ij}^k x_{ij}^k.$$

Definition 3. We call (2)-(4) the system of main constraints of the problem (1)-(6). Constraints (5)-(6) are the direct constraints of the problem (1)-(6).

Let us name, traditionally, different parts of the system of main constraints.

Definition 4. We call (2) the network part of the system of main constraints of the problem (1)-(6). Equations (3) are the additional part of the system of main constraints (additional constraints) of the problem (1)-(6). Inequalities (4) form the sparse part of the system of main constraints of the problem (1)-(6).

2. Example

In this section we introduce an example of the problem (1)-(6) for the multinetwork $G = (I,U), I = \{1,2,3,4\}$, $U = \{(1,3),(1,4), (2,1),(2,4),(3,2),(4,3)\}$. Let $K = \{1,2,3,4,5\}$ be the set of transported products (Figure 1). Characteristics of the structure of the network G are provided in Table 1. The mathematical model of the problem (1)-(6) for the multinetwork $G = (I,U)$ (Figure 1) can be represented as (1a)-(6a).

$$\begin{aligned} 2x_{13}^1 + x_{13}^2 + 3x_{13}^3 + 4x_{13}^5 + x_{14}^1 + 8x_{14}^4 + 5x_{21}^2 + x_{21}^4 + x_{24}^1 \\ +6x_{24}^2 + 2x_{24}^3 + 3x_{32}^2 + 2x_{32}^4 + x_{32}^5 + 4x_{43}^2 + x_{43}^3 \to \min, \end{aligned} \tag{1a}$$

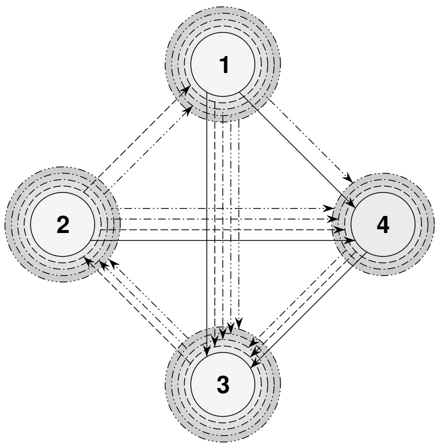

Figure 1: Multinetwork $G = (I, U)$

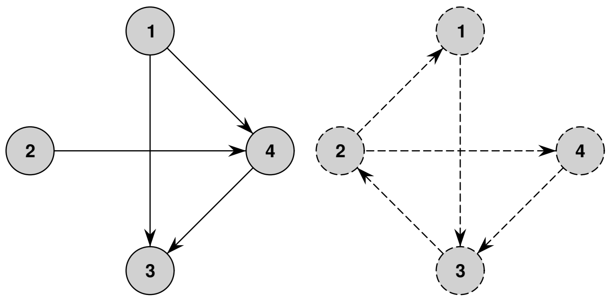

Figure 2: Networks $G^k = (I^k, U^k), k = 1, 2.$

$$\begin{array}{cc}
x^1_{13} + x^1_{14} = 1, & x^2_{13} - x^2_{21} = -0.75, \\
x^1_{24} = 0.5, & x^2_{21} + x^2_{24} - x^2_{32} = 0.25, \\
-x^1_{13} - x^1_{43} = -1, & x^2_{32} - x^2_{13} - x^2_{43} = 0.5, \\
x^1_{43} - x^1_{14} - x^1_{24} = -0.5, & x^2_{43} - x^2_{24} = 0, \\
x^3_{13} = 0, & x^4_{13} + x^4_{14} - x^4_{21} = 0.5, \\
x^3_{24} = 1, & x^4_{21} + x^4_{24} - x^4_{32} = -0.25, \\
-x^3_{13} - x^3_{43} = -1, & x^4_{32} - x^4_{13} = 0.75, \\
x^3_{43} - x^3_{24} = 0, & -x^4_{14} - x^4_{24} = -1, \\
\end{array}$$
$$\begin{array}{c}
x^5_{13} = 1.5, \\
-x^5_{32} = -0.5, \\
x^5_{32} - x^5_{13} = -1,
\end{array} \tag{2a}$$

(i,j)	$(1,3)$					$(1,4)$				
k	1	2	3	4	5	1	2	3	4	5
U^k	+	+	+	+	+	+			+	
U_0	+									
$K(i,j)$	$\{1,2,3,4,5\}$					$\{1,4\}$				
$K_1(i,j)$	$\{1,2,5\}$					$\{4\}$				
$K_0(i,j)$	$\{3,4\}$									
(i,j)	$(2,1)$					$(2,4)$				
k	1	2	3	4	5	1	2	3	4	5
U^k		+		+		+	+	+	+	
U_0										
$K(i,j)$	$\{2,4\}$					$\{1,2,3,4\}$				
$K_1(i,j)$	$\{2,4\}$					$\{1,3\}$				
$K_0(i,j)$										
(i,j)	$(3,2)$					$(4,3)$				
k	1	2	3	4	5	1	2	3	4	5
U^k		+		+	+	+	+	+		
U_0						+				
$K(i,j)$	$\{2,4,5\}$					$\{1,2,3\}$				
$K_1(i,j)$	$\{2,4,5\}$					$\{3\}$				
$K_0(i,j)$						$\{1,2\}$				

Table 1: Characteristics of the network structure

$$\begin{gathered}
2x_{13}^1 + 3x_{13}^2 + x_{13}^3 + x_{13}^4 + 2x_{13}^5 + x_{14}^1 \\
+3x_{14}^4 + 2x_{21}^2 + x_{21}^4 + x_{24}^1 + x_{24}^2 + 4x_{24}^3 \\
+x_{24}^4 + 2x_{32}^2 + 2x_{32}^4 + 3x_{32}^5 + x_{43}^1 + x_{43}^2 + 3x_{43}^3 = 42, \\
2x_{13}^1 + 2x_{13}^2 + 3x_{13}^3 + x_{13}^4 + 2x_{13}^5 + 3x_{14}^1 \\
+x_{14}^4 + 2x_{21}^2 + x_{21}^4 + x_{24}^1 + 2x_{24}^2 + 4x_{24}^3 \\
+3x_{24}^4 + 2x_{32}^2 + 4x_{32}^4 + x_{32}^5 + x_{43}^1 + 3x_{43}^2 + 3x_{43}^3 = 58,
\end{gathered} \tag{3a}$$

$$\begin{gathered}
x_{13}^3 + x_{13}^4 \leq 6,\ x_{13}^3 \geq 0,\ x_{13}^4 \geq 0, \\
x_{43}^1 + x_{43}^2 \leq 7,\ x_{43}^1 \geq 0,\ x_{43}^2 \geq 0,
\end{gathered} \tag{4a}$$

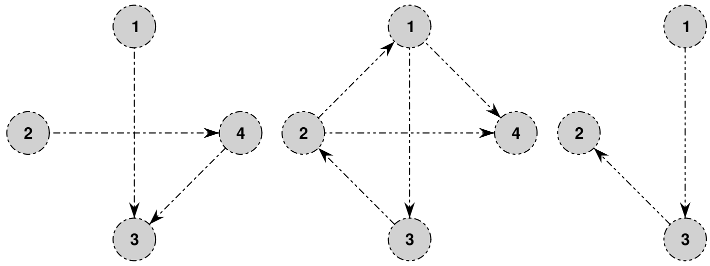

Figure 3: Networks $G^k = (I^k, U^k), k = 3, 4, 5.$

$$\begin{array}{llll} 0\leq x^1_{13}\leq 4, & 0\leq x^4_{14}\leq 4, & 0\leq x^1_{24}\leq 2, & 0\leq x^4_{32}\leq 3, \\ 0\leq x^2_{13}\leq 2, & 0\leq x^2_{21}\leq 5, & 0\leq x^3_{24}\leq 4, & 0\leq x^5_{32}\leq 2, \\ 0\leq x^5_{13}\leq 6, & 0\leq x^4_{21}\leq 2, & 0\leq x^2_{32}\leq 4, & 0\leq x^3_{43}\leq 4, \end{array} \tag{5a}$$

$$x^1_{14} \geq 0, \; x^2_{24} \geq 0, \; x^4_{24} \geq 0. \tag{6a}$$

3. Decomposition of the Network Support, Determinants of Cycles

In this section we define a support of the network $G = (I, U)$ for the problem (1)-(6). In order to give the definition we will need to introduce some more sets.

Let $U^k_S \subseteq U^k, k \in K$ be arbitrary subsets of arcs of the networks G^k. For each arc $(i, j) \in U$ we define a subset of products $K_S(i, j) \subseteq K(i, j)$, $K_S(i, j) = \{k \in K : (i, j)^k \in U^k_S\}$, transported through an arc (i, j). Also, for arcs from the set U_0 we introduce a subset U^*, such that $U^* \subseteq \overline{U}_0 \subseteq U_0$, $\overline{U}_0 = \big\{(i, j) \in U_0 : |K^0_S(i, j)| > 1\big\}$, where $K^0_S(i, j) = K_S(i, j) \bigcap K_0(i, j), (i, j) \in U_0$.

Let $U^k_S \subseteq U^k, k \in K$ be arbitrary subsets of arcs of the subnetworks G^k. Thus, for each arc $(i, j) \in U$ we can define a subset of products $K_S(i, j) \subseteq K(i, j)$, $K_S(i, j) = \{k \in K : (i, j)^k \in U^k_S\}$, transported through an arc (i, j). For arcs from the set U_0 we introduce a subset U^*, so that $U^* \subseteq U_0$. Also, we denote $K^0_S(i, j) = K_S(i, j) \bigcap K_0(i, j), (i, j) \in U_0$.

Definition 5. The aggregate of sets $U_S = \{U^k_S, k \in K; U^*\}$ $U^* \subseteq \overline{U}_0, \overline{U}_0 = \{(i, j) \in U_0 : |K^0_S(i, j)| > 1\}$, is a support of the network G (or, network support) for the problem (1)-(6), if the system

$$\sum_{j \in I^+_i(\hat{U}^k)} x^k_{ij} - \sum_{j \in I^-_i(\hat{U}^k)} x^k_{ji} = 0, \; i \in I^k, \; k \in K, \tag{7}$$

$$\sum_{k\in K}\sum_{(i,j)\in\hat{U}^k}\lambda_{ij}^{kp}x_{ij}^k=0,\ p=\overline{1,l}, \tag{8}$$

$$\sum_{k\in K_S^0(i,j)}x_{ij}^k=0,\ (i,j)\in\hat{U}^*, \tag{9}$$

has only the trivial solution $x_{ij}^k=0,(i,j)\in\hat{U}^k,k\in K$, when $\hat{U}^k=U_S^k$ and $\hat{U}^*=U^*$, $\hat{K}_S^0(i,j)=K_S^0(i,j)$, but has a non-trivial solution if either:

1) $\hat{U}^k=U_S^k,k\in K;\hat{U}^*=U^*\backslash(i_0,j_0),(i_0,j_0)\in U^*$, or

2) $\hat{U}^k=U_S^k,k\in K\backslash k_0,\hat{U}_S^{k_0}=U_S^{k_0}\bigcup(i_0,j_0)^{k_0},(i_0,j_0)^{k_0}\notin U_S^{k_0},\ k_0\in K;\hat{U}^*=U^*$,

$$\hat{K}_S^0(i,j)=\begin{cases}K_S^0(i,j), & \text{if}\quad (i,j)\neq(i_0,j_0),\\ K_S^0(i,j)\bigcup(K^0\bigcap K^0(i,j)), & \text{if}\quad (i,j)=(i_0,j_0).\end{cases}$$

Consider the support structure $U_S=\{U_S^k,k\in K;U^*\},U_S^k\subset U^k,k\in K;U^*\subset\overline{U}_0$, $\overline{U}_0=\{(i,j)\in U_0:|K_S^0|>1\},K_S(i,j)=\{k\in K(i,j):(i,j)^k\in U_S^k\},(i,j)\in U,K_S^0(i,j)=K_S(i,j)\bigcap K_0(i,j),\ (i,j)\in U_0$. Let the set U_S^k contain l_k arcs $(i,j)^k$ such that removing them from the set U_S^k gives the set U_T^k such that the network $(I(U_T^k),U_T^k)$ contains no cycles but every network $(I(U_T^k),U_T^k\bigcup(i,j)^k)$ contains a cycle. Let us denote $U_C^k=U_S^k\setminus U_T^k,k\in K$.

Definition 6. The elements of the set $U_C=\bigcup\limits_{k\in K}U_C^k$ are called cyclic arcs. The elements of the set $U_T=\bigcup\limits_{k\in K}U_T^k$ are called tree arcs.

Let K be the set $\{1,2,\ldots,|K|\}$. Let us suppose that U_S^k contain l_k independent cycles, $k\in K$. The cycles $Z_k=\{L_{\tau\rho}^k,(\tau,\rho)^k\in U_C^k\}$ of the network G are called independent if every cycle has at least one edge that does not belong to other cycles and the unitary circulations [3], [5] form a linearly-independent system of vectors, $k\in K$.

Every arc $(\tau,\rho)^k\in U_C^k$ belongs to some cycle $L_{\tau\rho}^k$ from the set U_S^k. If the network $G_S^k=(I(U_S^k),U_S^k)$ is connected then the cycles $Z_k=\{L_{\tau\rho}^k,(\tau,\rho)^k\in U_C^k\},k\in K$ introduced the same way form the fundamental set of cycles. We introduse arbitrary numbering of arcs within the set and $U_C=\bigcup\limits_{k\in K}U_C^k$. Let us mark every arc $(\tau\rho)^k\in U_C^k$ with the number $t=t(\tau\rho)^k$. Let us consider an arbitrary cycle $L_t^k,t=t(\tau,\rho)^k$. We choose the direction of cycle detour such that the arc $(\tau,\rho)^k$ is a forward one. Let L_t^{k+},L_t^{k-} be sets of forward and backward arcs of the cycle L_t^k correspondingly.

Definition 7. The number $R_p(L_t^k) = \sum\limits_{(i,j)^k \in L_t^k} \lambda_{ij}^{kp} \text{sign}(i,j)^{L_t^k}$ is called the determinant of the cycle L_t^k with respect to the additional restriction (3) with the number p, where

$$\text{sign}(i,j)^{L_t^k} = \begin{cases} 1, & \text{if} \quad (i,j)^k \in L_t^{k+}; \\ -1, & \text{if} \quad (i,j)^k \in L_t^{k-}; \\ 0, & \text{if} \quad (i,j)^k \notin L_t^k. \end{cases}$$

Let us put the arcs of the set U^* in arbitrary order. Let $\xi = \xi(i,j)$ be the serial number of the multiarc (i,j) in the set $U^*, 1 \leq \xi \leq m, m = |U^*|$. We build the matrix $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where $D_1 = (R_p(L_t^k), p = \overline{1,l}, t = \overline{1,\widetilde{t}}), \widetilde{t} = |U_c|$ is a matrix of order $l \times \widetilde{t}$ consisting of determinants $R_p(L_t^k) = R_p(L_{(\tau,\rho)}^k)$, $t = t(\tau,\rho)$ with respect to the restrictions (3), $D_2 = (\delta_{\xi(i,j)}(L_t^k), \xi(i,j) = \overline{1,m}, t = \overline{1,\widetilde{t}})$ is an $m \times \widetilde{t}$-matrix that consists of the following elements:

$$\delta_{\xi(i,j)}(L_t^k) = \begin{cases} 1, & \text{if } (i,j)^k \in L_{t(\tau,\rho)}^{k+}, k \in K_0(i,j), (i,j) \in U_0; \\ -1, & \text{if } (i,j)^k \in L_{t(\tau,\rho)}^{k-}, k \in K_0(i,j), (i,j) \in U_0; \\ 0, & \text{if } (i,j)^k \notin L_{t(\tau,\rho)}^{k}, k \in K_0(i,j), (i,j) \in U_0; \\ 0, & \text{if } (i,j)^k \notin L_{t(\tau,\rho)}^{k}, k \in K(i,j) \backslash K_0(i,j), \\ & \quad (i,j) \in U_0; \end{cases}$$

$$\xi = \xi(i,j), (i,j) \in U^*.$$

If $\widetilde{t} \neq l + m, m = |U^*|$ we supply the matrix D with zeros to make it a square matrix of order $\max\{\widetilde{t}, l + |U^*|\}$. Let $R(U_S) = \det D$.

4. Theoretical – Graphical Properties

Theorem 1. (Criterion of Support) *The set of arcs* $U_S = \{U_S^k, k \in K, U^*\}$, $U_S^k \subset U^k, k \in K; U^* \subset \overline{U}_0$, $\overline{U}_0 = \{(i,j) \in U_0 : |K_S^0| > 1\}, K_S(i,j) = \{k \in K(i,j) : (i,j)^k \in U_S^k\}, (i,j) \in U, K_S^0(i,j) = K_S(i,j) \bigcap K_0(i,j), (i,j) \in U_0$ *is a support of the network* G *iff the following conditions are met:*

1. $I(U_S^k) = I^k, k \in K$;
2. U_S^k *is a connected set,* $k \in K$;
3. $R(U_S) \neq 0$.

Proof. Is made [5]. □

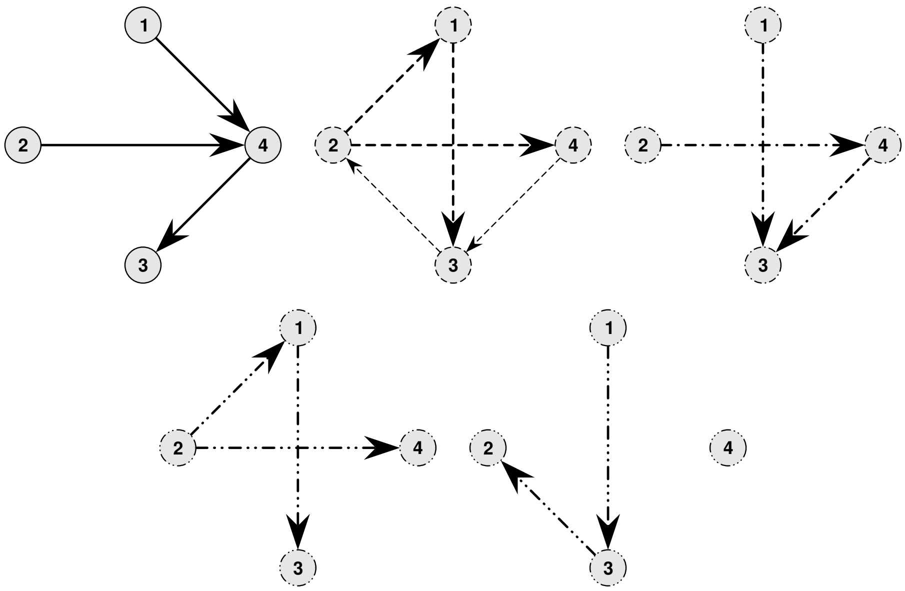

Figure 4: A support for the problem (1a)-(6a): bold arrows mean tree arcs, thin arrows mean cyclic arcs

(i,j)	$(1,3)$	$(1,4)$	$(2,1)$
$K_S(i,j)$	$\{2,3,4,5\}$	$\{1\}$	$\{2,4\}$
$K_S^1(i,j)$	$\{2,5\}$	$\varnothing$	$\{2,4\}$
$K_S^0(i,j)$	$\{3,4\}$		
$K_N(i,j)$	$\{1\}$	$\{4\}$	$\varnothing$
$K_N^1(i,j)$	$\{1\}$	$\{4\}$	$\varnothing$
$K_N^0(i,j)$	$\varnothing$		
(i,j)	$(2,4)$	$(3,2)$	$(4,3)$
$K_S(i,j)$	$\{1,2,3,4\}$	$\{2,5\}$	$\{2,1,3\}$
$K_S^1(i,j)$	$\{1,3\}$	$\{2,5\}$	$\{3\}$
$K_S^0(i,j)$			$\{2,1\}$
$K_N(i,j)$	$\varnothing$	$\{4\}$	$\varnothing$
$K_N^1(i,j)$	$\varnothing$	$\{4\}$	$\varnothing$
$K_N^0(i,j)$			$\varnothing$

Table 2: Characteristics of the support for the problem (1a)-(6a)

Let $D = D(U_S)$ be the matrix of determinants that corresponds to the support $U_S = \{U_S^k, k \in K, U^*\}$.

An example of a support for the problem (1a)-(6a) can be found at Figure 4. The characteristics of the support are represented in Table 2, where

$$K_S^1(i,j) = K_S(i,j) \bigcap K_1(i,j), (i,j) \in U;$$

$$K_S^0(i,j) = K_S(i,j) \bigcap K_0(i,j), (i,j) \in U_0;$$

$$K_N(i,j) = K(i,j) \setminus K_S(i,j);$$

$$K_N^1(i,j) = K_N(i,j) \bigcap K_1(i,j), (i,j) \in U;$$

$$K_N^0(i,j) = K_N(i,j) \bigcap K_0(i,j), (i,j) \in U_0.$$

$$U^* = \varnothing,\ D = \begin{bmatrix} R_1(L_{32}^2) & R_1(L_{43}^2) \\ R_2(L_{32}^2) & R_2(L_{43}^2) \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 6 & 1 \end{bmatrix},\ \det D \neq 0.$$

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